University of Notre Dame
Department of Electrical Engineering September 14, 2007
EE 67003: Advanced Digital Signal Processing Dr. A. Zaidi

Homework 1: Solutions

Problems from textbook:

Problem 6.1: (20 points)

$F = \Omega/(2\pi)$

(a) $dx_a(t)/dt \circ\!\!-\!\!\bullet\, \hat{X}_a(F) = j2\pi F\, X_a(F)$, then $F_s = 2B$

(b) $x_a^2(t) \circ\!\!-\!\!\bullet\, \hat{X}_a(F) = \hat{X}_a(F) = X_a(F) * X_a(F)$, then $F_s = 4B$

(c) $x_a(2t) \circ\!\!-\!\!\bullet\, \hat{X}_a(F) = 2\, X_a(F/2)$, then $F_s > 4B$

(d) $x_a(t)\, \cos(6\pi Bt) \circ\!\!-\!\!\bullet\, \hat{X}_a(F) = 0.5\, X_a(F+3B) + 0.5\, X_a(F-3B)$, resulting in $F_L = 2B$ and $F_H = 4B$. Hence, $F_s = 4B$.

(e) $x_a(t)\, \cos(7\pi Bt) \circ\!\!-\!\!\bullet\, \hat{X}_a(F) = 0.5\, X_a(F + 3.5B) + 0.5\, X_a(F - 3.5B)$, resulting in $F_L = 5B/2$ and $F_H = 9B/2$. Hence, $k_{\max} = \lfloor F_H/B \rfloor$ and $F_s = 2\, F_H/k_{\max} = 9B/2$.

Problem 6.9: (20 points)

(a) $F = \Omega/(2\pi)$

$$\begin{aligned} x_a(t) &= e^{-j2\pi F_0 t} \cdot u(t) \\ X_a(F) &= \int_0^\infty x_a(t)\, e^{-j2\pi F t}\, dt \\ &= \int_0^\infty e^{-j2\pi F_0 t} e^{-j2\pi F t}\, dt \\ &= \int_0^\infty e^{-j2\pi (F+F_0) t}\, dt \\ &= \left. \frac{e^{-j2\pi(F+F_0)t}}{-j2\pi(F+F_0)} \right|_0^\infty \\ X_a(F) &= \frac{1}{j2\pi(F+F_0)} + \frac{1}{2}\delta(F+F_0) \end{aligned}$$

(b) $\omega = 2\pi f$, $f = F/F_s$

$$\begin{aligned} x(n) &= e^{-j2\pi n F_0/F_s} \cdot u(n) \\ X(f) &= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} \\ &= \sum_{n=0}^{\infty} e^{-j2\pi n F_0/F_s} e^{-j2\pi f n} \\ &= \sum_{n=0}^{\infty} e^{-j2\pi n (f + F_0/F_s)} \\ &= \frac{1}{1 - e^{-j2\pi(f+F_0/F_s)}} + \pi \sum_{\lambda=-\infty}^{\infty} \delta(2\pi(f + F_0/F_s) + 2\pi\lambda) \end{aligned}$$

(with $u(n) \circ\!\!-\!\!\bullet\ \frac{1}{1-e^{-j\omega}} + \pi \sum_{\lambda=-\infty}^{\infty} \delta(\omega + 2\pi\lambda)$)

(e) Signi£cant aliasing occurs at $F_s = 10\,\text{Hz}$.

Problem 1: (20 points)

(a) Since there is no aliasing involved in this process, we may choose T to be any value (D/A converter). Choose $T = 1$ for simplicity $\Rightarrow X_a(j\Omega) = 0$

for $|\Omega| \geq \frac{\pi}{T}$. Since $Y_a(j\Omega) = H_a(j\Omega) \cdot V_a(j\Omega)$, $Y_a(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$. Therefore, there will be no aliasing problems in going from $y_a(t)$ to $y(n)$.

Recall that the relationship $\Omega = \omega/T$. We can simply use this in our system conversion for $T = 1$:

$$H(e^{j\omega}) = e^{-j\omega/2}$$
$$H_a(j\Omega) = e^{-j\Omega T/2} = e^{-j\Omega/2}.$$

Note that the choice of T and therefore $H_a(j\Omega)$ is not unique.

(b)

$$\begin{aligned}\cos\left(\frac{5}{2}\pi n - \frac{\pi}{4}\right) &= \frac{1}{2}\left[e^{j(\frac{5}{2}\pi n - \frac{\pi}{4})} + e^{-j(\frac{5}{2}\pi n - \frac{\pi}{4})}\right] \\ &= \frac{1}{2}e^{j\frac{5}{2}\pi n}\, e^{-j\frac{\pi}{4}} + \frac{1}{2}e^{-j\frac{5}{2}\pi n}\, e^{j\frac{\pi}{4}}\end{aligned}$$

Since $H(e^{j\Omega})$ is an LTI system, we can £nd the response to each of the two eigenfunctions separately:

$$y(n) = \frac{1}{2}\, e^{-j\frac{\pi}{4}}\, H(e^{j\frac{5}{2}\pi})\, e^{j\frac{5}{2}\pi n} + \frac{1}{2}\, e^{j\frac{\pi}{4}}\, H(e^{-j\frac{5}{2}\pi})\, e^{-j\frac{5\pi}{2}n}.$$

Since $H(e^{j\omega})$ is de£ned for $0 \leq |\omega| \leq \pi$ we must evaluate the frequency at the baseband, $5\pi/2 \Rightarrow 5\pi/2 - 2\pi = \pi/2$. Therefore, $H(e^{j\frac{5}{2}\pi}) = e^{-j\frac{\pi}{4}}$, $H(e^{-j\frac{5}{2}\pi}) = e^{j\frac{\pi}{4}}$ and

$$y(n) = \frac{1}{2}\left[e^{j(\frac{5}{2}\pi n - \frac{\pi}{2})} + e^{-j(\frac{5}{2}\pi n - \frac{\pi}{2})}\right] = \cos\left(\frac{5}{2}\pi n - \frac{\pi}{2}\right).$$

Problem 2: (40 points)

(a) $y_{a1}(t) = 1/T \cdot y_{a2}(t)$ (constant factor): Convolution is a linear process, aliasing is a linear process. Periodic convolution is equivalent to convolution followed by aliasing.

$y_{a1}(t) \neq x_a^2(t)$: System 2 at step 1 shows $\mathcal{F}\{x_a^2(t)\}$. This is clearly not $\mathcal{F}\{y_{a1}(t)\}$. $\mathcal{F}\{y_{a1}(t)\}$ is an aliased version of $\mathcal{F}\{x_a(t)\}$.

(b) $y_{a1}(t) \neq x_a^2(t)$ for the same reason as in part (a).

(c)

$$
\begin{aligned}
x_a(t) &= A\,\cos(2\pi\,15\,t)\\
x_a^3(t) &= 3/4\cdot A^3\cdot\cos(2\pi\,15\,t) + 1/4\cdot A^3\cdot\cos(3\cdot 2\pi\,15\,t)) = y_a(t)\\
y(n) &= x^3(n) = 3/4\,A^3\cdot\cos(2\pi\,15\,nT) + 1/4\cdot A^3\cdot\cos(3\cdot 2\pi\,15\,nT)), \quad T = 1/f_s\\
y(n) &= 3/4\,A^3\cdot\cos\left(\frac{3}{4}\pi\,n\right) + 1/4\cdot A^3\cdot\cos\left(\frac{1}{4}\pi\,n\right)\\
y(n) &= x^3(n) \quad\Rightarrow\quad y_1(n) = x(n).
\end{aligned}
$$

(d) This is the inverse part to (c). Since multiplication in time corresponds to convolution in frequency, a signal $x_a^2(t)$ has at most twice the bandwidth of $x_a(t)$. Therefore, $x_a^{1/2}(t)$ will have at least half the bandwodth of $x_a(t)$. If we run our signal through a box that will raise it to the $1/M$-th power, then the sampling frequency can be decreased by a factor of M.