Proximal methods for convex minimization of φ-divergences. Application to computer vision.

Mireille El Gheche

Supervised by Jean-Christophe Pesquet and Joumana Farah

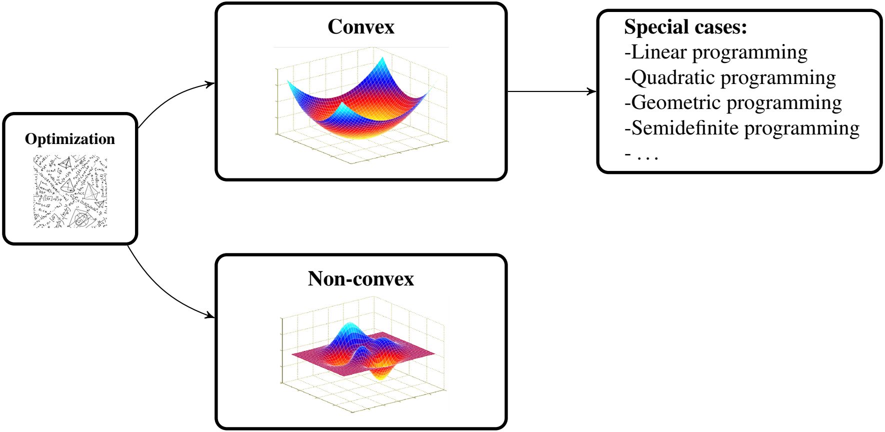
University of Paris-Est Lebanese University

27 May 2014

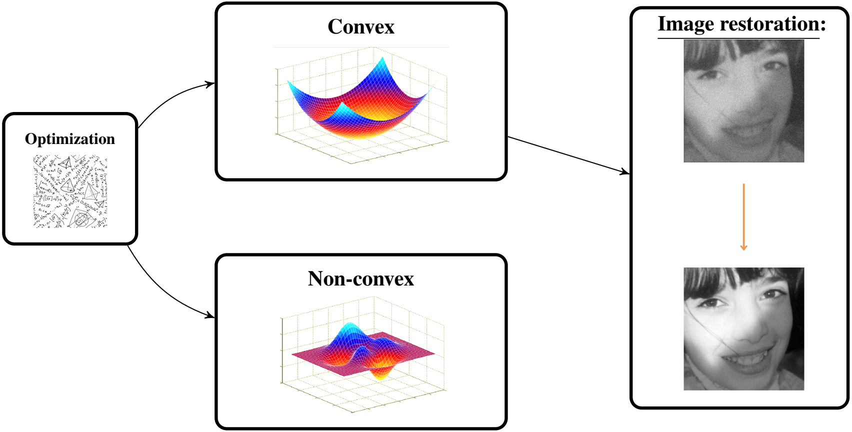
UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

Lebanese University Doctoral school

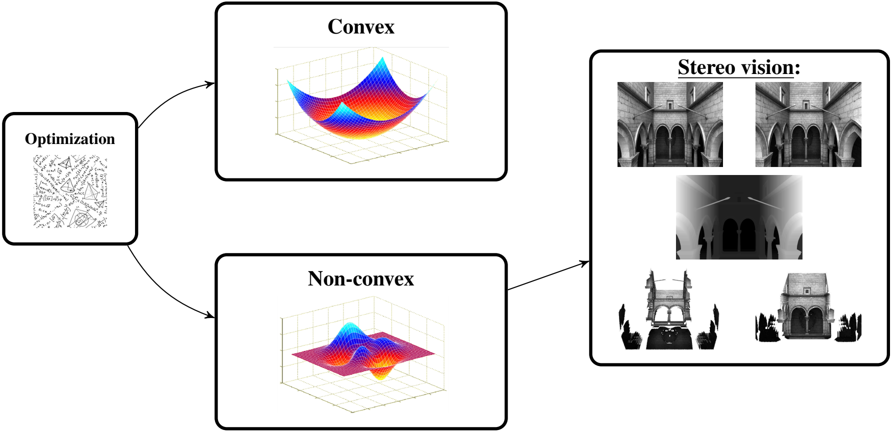
OVERVIEW


Optimization
Convex
Non-convex
Special cases:
-Linear programming
-Quadratic programming
-Geometric programming
-Semidefinite programming
- . . .

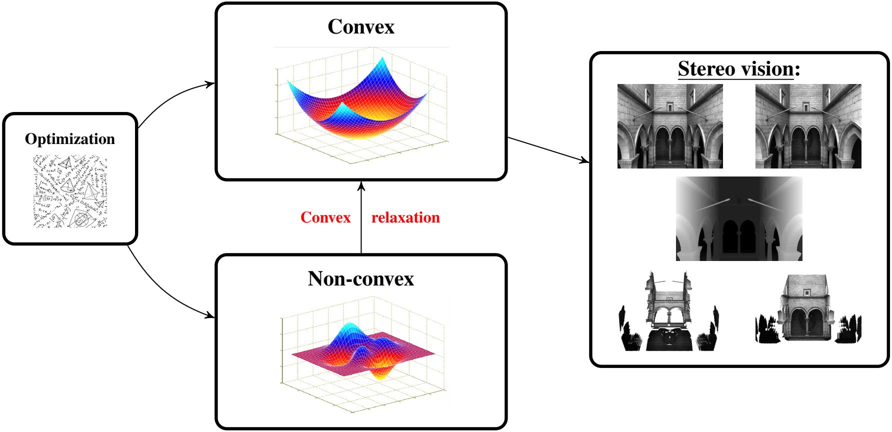
OVERVIEW



OVERVIEW



OVERVIEW


Optimization
Convex
Non-convex
Convex relaxation
Stereo vision:

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms
2. Can we effectively solve nonconvex optimization problems via convex relaxation ?

Challenges

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms
2. Can we effectively solve nonconvex optimization problems via convex relaxation ?
   - Stereo vision

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms
2. Can we effectively solve nonconvex optimization problems via convex relaxation ?
   - Stereo vision
   - multi-view

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms
2. Can we effectively solve nonconvex optimization problems via convex relaxation ?
   - Stereo vision
   - multi-view
3. Which kind of measures constitute appropriate cost functions ?

CHALLENGES

1. How to efficiently solve large-size non-smooth convex optimization problems?
   - Splitting proximal methods
   - Primal-dual algorithms
2. Can we effectively solve nonconvex optimization problems via convex relaxation ?
   - Stereo vision
   - multi-view
3. Which kind of measures constitute appropriate cost functions ?
   - φ-divergences

CONTENTS

PUBLICATIONS

Journal

- C. Chaux, M. El Gheche, J. Farah, J.-C. Pesquet, and B. Pesquet- Popescu, A parallel proximal splitting method for disparity estimation from multicomponent images under illumination variation, in Journal of Mathematical Imaging and Vision, pages 1-12, 10.1007/s10851-012-0361-z, 2012.

Journals in preparation:

- M. El Gheche et al., Proximity operators of some discrete information divergences, *in preparation*.
- M. El Gheche et al., Multi-view disparity estimation using a robust multi-label technique, *in preparation*.

Conferences (5):

- M. El Gheche, A. Jezierska, J.-C. Pesquet and J. Farah, A proximal approach for signal recovery based on information measures, in EUSIPCO 2013.
- M. El Gheche, J.-C. Pesquet and J. Farah, A proximal approach for optimization problems involving Kullback divergences, in ICASSP 2013, Vancouver, Canada, 26-31 May 2013.
- M. El Gheche, C. Chaux, J.-C. Pesquet, J. Farah and B. Pesquet-Popescu, Disparity map estimation under convex constraints using proximal algorithms, in SIPS 2011, Pages 293-298, Beirut, Lebanon, 4-7 Oct. 2011.
- M. El Gheche, J.-C. Pesquet, C. Chaux, J. Farah et B. Pesquet-Popescu, Méthodes proximales pour l'estimation du champ de disparité à partir d'une paire d'images stéréoscopiques en présence de variations d'illumination, GRETSI 2011, Bordeaux, France, 5-8 Sep. 2011.
- M. El Gheche, J.-C. Pesquet, J. Farah, M. Kaaniche and B. Pesquet-Popescu, Proximal splitting methods for depth estimation, in ICASSP 2011, Pages 853-856, Prague, Czech Republic, 22-27 May 2011.

PROXIMAL ALGORITHMS

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x) + g(Lx) + h(x)$$

PROXIMAL ALGORITHMS

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x)+g(Lx)+h(x)$$

- $f\in\Gamma_0(\mathbb{R}^N)$
  $\Gamma_0(\mathbb{R}^N)$: set of convex l.s.c. proper functions from $\mathbb{R}^N$ to $]-\infty,+\infty]$

PROXIMAL ALGORITHMS

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x) + g(Lx) + h(x)$$

- $f \in \Gamma_0(\mathbb{R}^N)$
  $\Gamma_0(\mathbb{R}^N)$: set of convex l.s.c. proper functions from $\mathbb{R}^N$ to $]-\infty, +\infty]$
- $L \in \mathbb{R}^{M\times N}$ and $g \in \Gamma_0(\mathbb{R}^M)$

PROXIMAL ALGORITHMS

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x)+g(Lx)+h(x)$$

- $f \in \Gamma_0(\mathbb{R}^N)$
  $\Gamma_0(\mathbb{R}^N)$: set of convex l.s.c. proper functions from $\mathbb{R}^N$ to $]-\infty,+\infty]$
- $L \in \mathbb{R}^{M\times N}$ and $g \in \Gamma_0(\mathbb{R}^M)$
- $h\colon \mathbb{R}^N \mapsto\,]-\infty,+\infty]$ differentiable with ∇h Lipschitzian

PROXIMAL ALGORITHMS

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x)+g(Lx)+h(x)$$

- $f \in \Gamma_0(\mathbb{R}^N)$
  $\Gamma_0(\mathbb{R}^N)$: set of convex l.s.c. proper functions from $\mathbb{R}^N$ to $]-\infty,+\infty]$
- $L \in \mathbb{R}^{M\times N}$ and $g \in \Gamma_0(\mathbb{R}^M)$
- $h\colon \mathbb{R}^N \mapsto]-\infty,+\infty]$ differentiable with ∇h Lipschitzian

Possible choices for f and g:

- non-smooth functions (e.g. $\ell_{q,p}$-norm, max, ...)

Proximal algorithms

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \quad f(x) + g(Lx) + h(x)$$

- $f \in \Gamma_0(\mathbb{R}^N)$
  $\Gamma_0(\mathbb{R}^N)$: set of convex l.s.c. proper functions from $\mathbb{R}^N$ to $]-\infty, +\infty]$
- $L \in \mathbb{R}^{M\times N}$ and $g \in \Gamma_0(\mathbb{R}^M)$
- $h\colon \mathbb{R}^N \mapsto]-\infty, +\infty]$ differentiable with ∇h Lipschitzian

Possible choices for f and g:

- non-smooth functions (e.g. $\ell_{q,p}$-norm, max, ...)
- indicator function of a closed convex subset $C \subset \mathbb{R}^N$

$$\iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

PROXIMAL ALGORITHMS

- Proximity operator of f

$$(\forall x \in \mathbb{R}^N) \quad \mathrm{prox}_f(x) = \mathrm{argmin}_{u \in \mathbb{R}^N} \; \frac{1}{2}\|u - x\|^2 + f(u)$$

PROXIMAL ALGORITHMS

- Proximity operator of f

$$(\forall x \in \mathbb{R}^N) \quad \mathrm{prox}_f(x) = \mathrm{argmin}_{u \in \mathbb{R}^N} \ \frac{1}{2}\|u - x\|^2 + f(u)$$

- Projection onto C

$$(\forall x \in \mathbb{R}^N) \quad P_C(x) = \mathrm{prox}_{\iota_C}(x) = \mathrm{argmin}_{u \in C} \ \frac{1}{2}\|u - x\|^2$$

PROXIMAL ALGORITHMS

Parallel ProXimal Algorithm (PPXA+ [Pesquet & Pustelnik 2011])

$$f = h = 0$$
$$\gamma > 0, \lambda \in]0, 2[$$
$$(x_0, v_0) \in \mathbb{R}^N \times \mathbb{R}^M$$

For $n = 0, 1, \ldots$

$$\widetilde{x}_n = \mathrm{prox}_{\gamma g}(v_n)$$
$$\widehat{x}_n = (L^\top L)^{-1} L^\top \widetilde{x}_n$$
$$v_{n+1} = v_n + \lambda\Big(L(2\widehat{x}_n - x_n) - \widetilde{x}_n\Big)$$
$$x_{n+1} = x_n + \lambda(\widehat{x}_n - x_n)$$

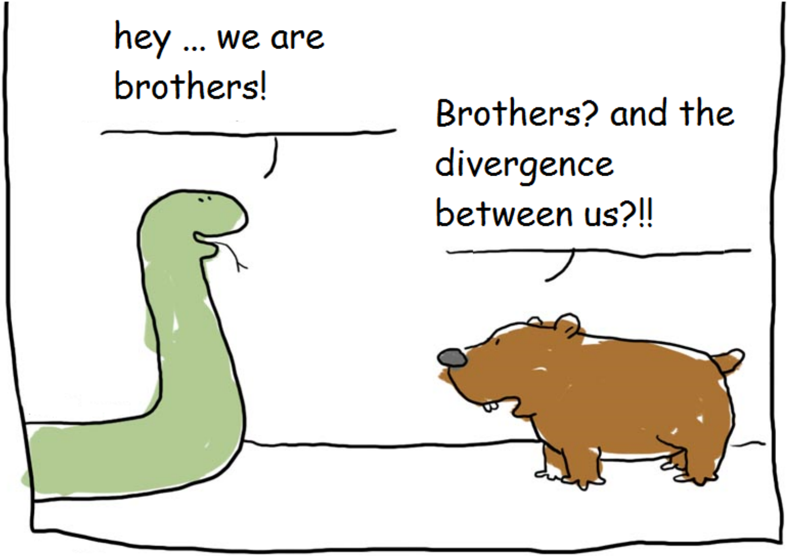
PROXIMAL ALGORITHMS

Montone+Lipschitz Forward Backward Forward
(M+L FBF [Combettes & Pesquet 2011])

$$\gamma \in \left]0, (\beta + \|L\|)^{-1}\right[$$
$$(x_0, v_0) \in \mathbb{R}^N \times \mathbb{R}^M$$

For $n = 0, 1, \ldots$

$$\widehat{x}_n = L^\top v_n + \nabla h(x_n)$$
$$\widehat{v}_n = L x_n$$
$$\widetilde{x}_n = \mathrm{prox}_{\gamma f}(x_n - \gamma \widehat{x}_n)$$
$$\widetilde{v}_n = \mathrm{prox}_{\gamma g^*}(v_n + \gamma \widehat{v}_n)$$
$$x_{n+1} = \widetilde{x}_n - \gamma\big(L^\top \widetilde{v}_n - \widehat{x}_n + \nabla h(\widetilde{x}_n)\big)$$
$$v_{n+1} = \widetilde{v}_n + \gamma(L\widetilde{x}_n - \widehat{v}_n)$$


hey ... we are brothers!
Brothers? and the divergence between us?!!

Objective

General formulation

$$\boxed{\underset{x\in\mathbb{R}^N}{\text{minimize}}\;\; D(Ax,Bx) + \sum_{s=1}^{S} R_s(L_s x)}$$

where

- $A, B \in \mathbb{R}^{P\times N}$
- $D \in \Gamma_0(\mathbb{R}^P \times \mathbb{R}^P)$
- $\forall s \in \{1,\ldots,S\}$, $L_s \in \mathbb{R}^{K_s\times N}$ and $R_s \in \Gamma_0(\mathbb{R}^{K_s})$

OBJECTIVE

Particular case

$$\underset{(p,q)\in(\mathbb{R}^P)^2}{\text{minimize}} \; D(p,q) + \sum_{s=1}^{S} R_s(U_s p + V_s q)$$

where

- $x = [p^\top \;\; q^\top]^\top$, with $p = (p^{(i)})_{1\le i\le P}$ and $q = (q^{(i)})_{1\le i\le P}$
- $A = [I \;\; 0]$ and $B = [0 \;\; I]$
- $(\forall s \in \{1,\dots,S\})$, $L_s = [U_s \;\; V_s]$ and $U_s, V_s \in (\mathbb{R}^{K_s\times P})^2$

SCOPE OF PRIMAL-DUAL PROXIMAL ALGORITHMS

1. separable case: $D(p,q) = \sum_{i=1}^{P} \phi_1^{(i)}(p^{(i)}) + \phi_2^{(i)}(q^{(i)})$

$(\forall i \in \{1,\dots,P\})\ \ \phi_1^{(i)}, \phi_2^{(i)} \in \Gamma_0(\mathbb{R})$

SCOPE OF PRIMAL-DUAL PROXIMAL ALGORITHMS

1. separable case: $D(p,q) = \sum_{i=1}^{P} \phi_1^{(i)}(p^{(i)}) + \phi_2^{(i)}(q^{(i)})$

   $(\forall i \in \{1,\ldots,P\})\ \ \phi_1^{(i)}, \phi_2^{(i)} \in \Gamma_0(\mathbb{R})$
2. non-separable case:

SCOPE OF PRIMAL-DUAL PROXIMAL ALGORITHMS

1. separable case: $D(p,q) = \sum_{i=1}^{P} \phi_1^{(i)}(p^{(i)}) + \phi_2^{(i)}(q^{(i)})$

   $(\forall i \in \{1,\dots,P\}) \;\; \phi_1^{(i)}, \phi_2^{(i)} \in \Gamma_0(\mathbb{R})$

2. non-separable case:

   - $D(p,q) = \sum_{i=1}^{P} \phi^{(i)}(\alpha\, p^{(i)} + \beta\, q^{(i)})$

     $(\forall i \in \{1,\dots,P\}) \;\; \phi^{(i)} \in \Gamma_0(\mathbb{R})$ and $(\alpha,\beta) \in \mathbb{R}^2$

Scope of primal-dual proximal algorithms

1. separable case: $D(p,q) = \sum_{i=1}^{P} \phi_1^{(i)}(p^{(i)}) + \phi_2^{(i)}(q^{(i)})$

   $(\forall i \in \{1,\ldots,P\})\ \ \phi_1^{(i)}, \phi_2^{(i)} \in \Gamma_0(\mathbb{R})$

2. non-separable case:
   - $D(p,q) = \sum_{i=1}^{P} \phi^{(i)}(\alpha\, p^{(i)} + \beta\, q^{(i)})$
     $(\forall i \in \{1,\ldots,P\})\ \ \phi^{(i)} \in \Gamma_0(\mathbb{R})$ and $(\alpha,\beta) \in \mathbb{R}^2$
   - $D = \iota_C$ with C being a closed convex subset of $\mathbb{R}^{2P}$

SCOPE OF PRIMAL-DUAL PROXIMAL ALGORITHMS

1. separable case: $D(p,q) = \sum_{i=1}^{P} \phi_1^{(i)}(p^{(i)}) + \phi_2^{(i)}(q^{(i)})$

   $(\forall i \in \{1,\ldots,P\})\ \ \phi_1^{(i)}, \phi_2^{(i)} \in \Gamma_0(\mathbb{R})$

2. non-separable case:
   - $D(p,q) = \sum_{i=1}^{P} \phi^{(i)}(\alpha\, p^{(i)} + \beta\, q^{(i)})$
     $(\forall i \in \{1,\ldots,P\})\ \ \phi^{(i)} \in \Gamma_0(\mathbb{R})$ and $(\alpha,\beta) \in \mathbb{R}^2$
   - $D = \iota_C$ with C being a closed convex subset of $\mathbb{R}^{2P}$
   - $D = \phi \circ d_C$ with $d_C = \|\cdot - P_C \cdot\|$
     $\phi \in \Gamma_0(\mathbb{R})$

MOTIVATIONS

Additive information measures:

$$D(p,q) = \sum_{i=1}^{P} \Phi(p^{(i)}, q^{(i)})$$

where

$$(\forall (\upsilon,\xi) \in \mathbb{R}^2) \quad \boxed{\Phi(\upsilon,\xi) = \begin{cases} \xi\varphi\left(\frac{\upsilon}{\xi}\right) & \text{if } \upsilon \in [0,+\infty[\text{ and } \xi \in]0,+\infty[\\ \upsilon \lim\limits_{\zeta\to+\infty} \frac{\varphi(\zeta)}{\zeta} & \text{if } \upsilon \in]0,+\infty[\text{ and } \xi = 0 \\ 0 & \text{if } \upsilon = \xi = 0 \\ +\infty & \text{otherwise} \end{cases}}$$

and $\varphi \in \Gamma_0(\mathbb{R})$, $\varphi\colon \mathbb{R} \to [0,+\infty]$ is twice differentiable on $]0,+\infty[$.

RELATED WORK

Optimization problems involving information measures:

- ✗ One of the two variables is fixed [Byrne 1993], [Richardson 1972], [Dupé et al. 2009], [Pustelnik et al. 2011], [Steidl et al. 2012]
- ✗ Alternating minimization [Blahut 1972], [Arimoto 1972], [Bauschke 2011]

RELATED WORK

Optimization problems involving information measures:

- ✗ One of the two variables is fixed [Byrne 1993], [Richardson 1972], [Dupé et al. 2009], [Pustelnik et al. 2011], [Steidl et al. 2012]
- ✗ Alternating minimization [Blahut 1972], [Arimoto 1972], [Bauschke 2011]

Contributions:

- ✓ Proximity operator of two-variable convex functions
- ✓ General form of optimization problems
- ✓ Application to image restoration

EXAMPLES OF φ-DIVERGENCES [BASSEVILLE 2010]

- Kullback-Leibler : $\varphi(\zeta) = \zeta \ln \zeta - \zeta + 1$

$$\Phi\colon (\upsilon,\xi) \mapsto \begin{cases} \upsilon \ln\left(\frac{\upsilon}{\xi}\right) + \xi - \upsilon & \text{if } (\upsilon,\xi) \in \left]0,+\infty\right[^2 \\ \xi & \text{if } \upsilon = 0 \text{ and } \xi \in \left[0,+\infty\right[\\ +\infty & \text{otherwise.} \end{cases}$$

EXAMPLES OF φ-DIVERGENCES [BASSEVILLE 2010]

- **Kullback-Leibler** : $\varphi(\zeta) = \zeta \ln \zeta - \zeta + 1$
- **Jeffreys-Kullback** : $\varphi(\zeta) = (\zeta - 1) \ln \zeta$

$$\Phi\colon (\upsilon, \xi) \mapsto \begin{cases} (\upsilon - \xi)\big(\ln \upsilon - \ln \xi\big) & \text{if } (\upsilon, \xi) \in \,]0, +\infty[^2 \\ 0 & \text{if } \upsilon = \xi = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

EXAMPLES OF φ-DIVERGENCES [BASSEVILLE 2010]

- Kullback-Leibler : $\varphi(\zeta) = \zeta \ln \zeta - \zeta + 1$
- Jeffreys-Kullback : $\varphi(\zeta) = (\zeta - 1) \ln \zeta$
- Hellinger : $\varphi(\zeta) = \zeta + 1 - 2\sqrt{\zeta}$

$$\Phi\colon (\upsilon, \xi) \mapsto \begin{cases} (\sqrt{\upsilon} - \sqrt{\xi})^2 & \text{if } (\upsilon, \xi) \in [0, +\infty[^2 \\ +\infty & \text{otherwise.} \end{cases}$$

Examples of φ-divergences [Basseville 2010]

- Kullback-Leibler : $\varphi(\zeta) = \zeta \ln \zeta - \zeta + 1$
- Jeffreys-Kullback : $\varphi(\zeta) = (\zeta - 1) \ln \zeta$
- Hellinger : $\varphi(\zeta) = \zeta + 1 - 2\sqrt{\zeta}$
- Chi square : $\varphi(\zeta) = (\zeta - 1)^2$

$$\Phi \colon (\upsilon, \xi) \mapsto \begin{cases} \dfrac{(\upsilon - \xi)^2}{\xi} & \text{if } \upsilon \in [0, +\infty[\text{ and } \xi \in]0, +\infty[\\ 0 & \text{if } \upsilon = \xi = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

EXAMPLES OF φ-DIVERGENCES [BASSEVILLE 2010]

- Kullback-Leibler : $\varphi(\zeta) = \zeta \ln \zeta - \zeta + 1$
- Jeffreys-Kullback : $\varphi(\zeta) = (\zeta - 1) \ln \zeta$
- Hellinger : $\varphi(\zeta) = \zeta + 1 - 2\sqrt{\zeta}$
- Chi square : $\varphi(\zeta) = (\zeta - 1)^2$
- $I_\alpha, \alpha \in]0,1[$: $\varphi(\zeta) = 1 - \alpha + \alpha\zeta - \zeta^\alpha$

$$\Phi\colon (\upsilon, \xi) \mapsto \begin{cases} \alpha\upsilon + (1-\alpha)\xi - \upsilon^\alpha \xi^{1-\alpha} & \text{if } \upsilon \in [0,+\infty[\text{ and } \xi \in [0,+\infty[\\ +\infty & \text{otherwise} \end{cases}$$

DIVERGENCE PROXIMITY OPERATOR

Separability: $\big(\forall p = (p^{(i)})_{1\leq i\leq P} \in \mathbb{R}^P\big)\big(\forall q = (q^{(i)})_{1\leq i\leq P} \in \mathbb{R}^P\big)$

$$\boxed{D(p,q) = \sum_{i=1}^{P} \Phi(p^{(i)}, q^{(i)})}$$

Divergence Proximity operator

Separability: $\big(\forall p = (p^{(i)})_{1\le i\le P} \in \mathbb{R}^P\big)\big(\forall q = (q^{(i)})_{1\le i\le P} \in \mathbb{R}^P\big)$

$$\boxed{D(p,q) = \sum_{i=1}^{P} \Phi(p^{(i)}, q^{(i)})}$$

Proximity operator

$$(\forall \bar{p} = (\bar{p}^{(i)})_{1\le i\le P} \in \mathbb{R}^P)\ (\forall \bar{q} = (\bar{q}^{(i)})_{1\le i\le P} \in \mathbb{R}^P)$$

$$\operatorname{prox}_D(\bar{p},\bar{q}) = \Big(\operatorname{prox}_\Phi(\bar{p}^{(i)}, \bar{q}^{(i)})\Big)_{1\le i\le P}.$$

DIVERGENCE PROXIMITY OPERATOR

Let $\gamma \in \,]0, +\infty[$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$.

$$\boxed{\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \big(\overline{\upsilon} - \gamma\,\vartheta_-(\widehat{\zeta}), \overline{\xi} - \gamma\,\vartheta_+(\widehat{\zeta})\big)}$$

where $\widehat{\zeta} < \chi_+$ is the unique minimizer of strictly convex function ψ on $]\chi_-, +\infty[$.

DIVERGENCE PROXIMITY OPERATOR

Let $\gamma \in \,]0,+\infty[$ and $(\overline{\upsilon},\overline{\xi}) \in \mathbb{R}^2$.

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon},\overline{\xi}) = \big(\overline{\upsilon} - \gamma\,\vartheta_-(\widehat{\zeta}), \overline{\xi} - \gamma\,\vartheta_+(\widehat{\zeta})\big)$$

where $\widehat{\zeta} < \chi_+$ is the unique minimizer of strictly convex function ψ on $]\chi_-,+\infty[$.

- $\psi\colon\ ]0,+\infty[\,\to \mathbb{R}\colon \zeta \mapsto \zeta\varphi(\zeta^{-1}) - \Theta(\zeta) + \frac{\gamma^{-1}\overline{\upsilon}}{2}\zeta^2 - \gamma^{-1}\overline{\xi}\zeta$

where Θ denote a primitive of the function $\zeta \mapsto \zeta\varphi'(\zeta^{-1})$ on $]0,+\infty[$

- $\vartheta_-\colon\ ]0,+\infty[\,\to \mathbb{R}\colon \zeta \mapsto \varphi'(\zeta^{-1})$
- $\vartheta_+\colon\ ]0,+\infty[\,\to \mathbb{R}\colon \zeta \mapsto \varphi(\zeta^{-1}) - \zeta^{-1}\varphi'(\zeta^{-1})$

- $\chi_- = \inf\left\{\zeta \in \,]0,+\infty[\;\middle|\; \vartheta_-(\zeta) < \gamma^{-1}\overline{\upsilon}\right\}$
- $\chi_+ = \sup\left\{\zeta \in \,]0,+\infty[\;\middle|\; \vartheta_+(\zeta) < \gamma^{-1}\overline{\xi}\right\}$

DIVERGENCE PROXIMITY OPERATOR

Kullback-Leibler:

Let $\gamma > 0$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$,

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \begin{cases} \left(\overline{\upsilon} + \gamma \ln \widehat{\zeta}, \overline{\xi} + \gamma(\widehat{\zeta}^{-1} - 1)\right) & \text{if } \exp(\overline{\upsilon}/\gamma) > 1 - \gamma^{-1}\overline{\xi} \\ (0,0) & \text{otherwise} \end{cases}$$

where $\widehat{\zeta}$ is the minimizer on $]\exp(-\overline{\upsilon}/\gamma), +\infty[$ of

$$\psi(\zeta) = \left(\frac{\zeta^2}{2} - 1\right) \ln \zeta + \frac{1}{2}\left(\gamma^{-1}\overline{\upsilon} - \frac{1}{2}\right)\zeta^2 + (1 - \gamma^{-1}\overline{\xi})\zeta.$$

Divergence Proximity operator

Jeffrey-Kullback:

Let $\gamma > 0$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$,

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \begin{cases} \big(\overline{\upsilon} + \gamma\big(\ln \widehat{\zeta} + \widehat{\zeta} - 1\big), \overline{\xi} - \gamma\big(\ln \widehat{\zeta} - \widehat{\zeta}^{-1} + 1\big)\big) & \\ & \text{if } W(e^{1-\gamma^{-1}\overline{\upsilon}})W(e^{1-\gamma^{-1}\overline{\xi}}) < 1 \\ (0,0) & \text{otherwise} \end{cases}$$

where, $\widehat{\zeta}$ is the minimizer on $]W(e^{1-\gamma^{-1}\overline{\upsilon}}), +\infty[$ of

$$\psi(\zeta) = \Big(\frac{\zeta^2}{2} + \zeta - 1\Big) \ln \zeta + \frac{\zeta^3}{3} + \frac{1}{2}\Big(\gamma^{-1}\overline{\upsilon} - \frac{3}{2}\Big)\zeta^2 - \gamma^{-1}\overline{\xi}\zeta.$$

DIVERGENCE PROXIMITY OPERATOR

Hellinger:

Let $\gamma > 0$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \begin{cases} \left(\overline{\upsilon} + \gamma(\rho - 1), \overline{\xi} + \gamma\left(\frac{1}{\rho} - 1\right)\right) & \text{if } (\overline{\upsilon} < \gamma \;\text{ and } \\ & (1 - \gamma^{-1}\overline{\upsilon})(1 - \gamma^{-1}\overline{\xi}) < 1) \\ & \text{or } \overline{\upsilon} \geq \gamma \\ (0, 0) & \text{otherwise} \end{cases}$$

where ρ is the unique solution on $]\max(1 - \gamma^{-1}\overline{\upsilon}, 0), +\infty[$ of the equation:

$$\rho^4 + (\gamma^{-1}\overline{\upsilon} - 1)\rho^3 + (1 - \gamma^{-1}\overline{\xi})\rho - 1 = 0.$$

DIVERGENCE PROXIMITY OPERATOR

Chi-Square:

Let $\gamma > 0$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \begin{cases} \big(\overline{\upsilon} + 2\gamma(1-\rho), \overline{\xi} + \gamma(\rho^2 - 1)\big) & \text{if } \overline{\upsilon} > -2\gamma \\ & \text{and } \ \overline{\xi} > -\overline{\upsilon}\big(1 + (4\gamma)^{-1}\overline{\upsilon}\big) \\ \big(0, \max(\overline{\xi} - \gamma, 0)\big) & \text{otherwise} \end{cases}$$

where ρ is the unique solution on $]0, 1 + \gamma^{-1}\overline{\upsilon}/2[$ of

$$\rho^3 + (1 + \gamma^{-1}\overline{\xi})\rho = 2 + \gamma^{-1}\overline{\upsilon}.$$

DIVERGENCE PROXIMITY OPERATOR

I_α divergence:

Let $\gamma > 0$ and $(\overline{\upsilon}, \overline{\xi}) \in \mathbb{R}^2$

$$\operatorname{prox}_{\gamma\Phi}(\overline{\upsilon}, \overline{\xi}) = \begin{cases} \left(\overline{\upsilon} + \gamma\alpha(\widehat{\zeta}^{1-\alpha} - 1), \overline{\xi} + \gamma(1-\alpha)(\widehat{\zeta}^{-\alpha} - 1)\right) & \text{if } \Big(\overline{\upsilon} < \gamma\alpha \text{ and } \\ & \Big(1 - \frac{\gamma^{-1}\overline{\xi}}{(1-\alpha)}\Big) < \Big(1 - \frac{\overline{\upsilon}}{\gamma\alpha}\Big)^{\frac{\alpha}{\alpha-1}}\Big) \\ & \text{or } \overline{\upsilon} \geq \gamma\alpha \\ (0,0) & \text{otherwise} \end{cases}$$

where $\widehat{\zeta}$ is the unique solution on $\Big]\Big(\max\Big(1 - \frac{\overline{\upsilon}}{\gamma\alpha}, 0\Big)\Big)^{\frac{1}{1-\alpha}}, +\infty\Big[$ of

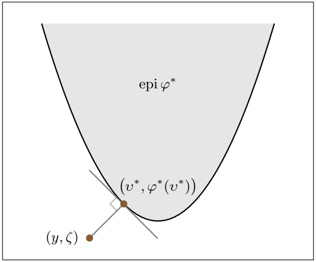
$$\alpha\widehat{\zeta}^2 + (\gamma^{-1}\overline{\upsilon} - \alpha)\widehat{\zeta}^{\alpha+1} + (1 - \alpha - \gamma^{-1}\overline{\xi})\widehat{\zeta}^{\alpha} = 1 - \alpha.$$

EPIGRAPHICAL PROJECTION

Let $\varphi^* \in \Gamma_0(\mathbb{R})$ the Fenchel-conjugate function of the restriction of φ on $[0, +\infty[$ and Φ is the perspective function of φ on $[0, +\infty[\, \times\,]0, +\infty[$.

The epigraph of φ^* is given by

$$(\forall (\upsilon^*, \xi^*) \in \mathbb{R}^2) \qquad \operatorname{epi} \varphi^* = \{(\upsilon^*, \xi^*) \in \mathbb{R}^2 \mid \varphi^*(\upsilon^*) \leq \xi^*\}$$



$\Rightarrow$ useful tool for splitting complex convex constraints

EPIGRAPHICAL PROJECTION

Let $\varphi^* \in \Gamma_0(\mathbb{R})$ the Fenchel-conjugate function of the restriction of φ on $[0, +\infty[$ and Φ is the perspective function of φ on $[0, +\infty[\times]0, +\infty[$.

The projection onto epi φ^* is given by

$$\big(\forall (y, \zeta) \in \mathbb{R}^2\big) \qquad \mathsf{P}_{\mathrm{epi}\,\varphi^*}(y, \zeta) = (y, -\zeta) - \mathrm{prox}_{\Phi}(y, -\zeta).$$

Epigraphical projection

Let $\varphi^* \in \Gamma_0(\mathbb{R})$ the Fenchel-conjugate function of the restriction of φ on $[0,+\infty[$ and Φ is the perspective function of φ on $[0,+\infty[\,\times\,]0,+\infty[$.

The projection onto epi φ^* is given by

$$\big(\forall (y,\zeta) \in \mathbb{R}^2\big) \qquad \mathsf{P}_{\mathrm{epi}\,\varphi^*}(y,\zeta) = (y,-\zeta) - \mathrm{prox}_{\Phi}(y,-\zeta).$$

Kullback-Leibler

$$\varphi^*(\zeta^*) = e^{\zeta^*} - 1$$

Epigraphical projection

Let $\varphi^* \in \Gamma_0(\mathbb{R})$ the Fenchel-conjugate function of the restriction of φ on $[0,+\infty[$ and Φ is the perspective function of φ on $[0,+\infty[\,\times\,]0,+\infty[$.

The projection onto epi φ^* is given by

$$\big(\forall (y,\zeta)\in\mathbb{R}^2\big) \qquad \mathsf{P}_{\operatorname{epi}\varphi^*}(y,\zeta) = (y,-\zeta) - \operatorname{prox}_{\Phi}(y,-\zeta).$$

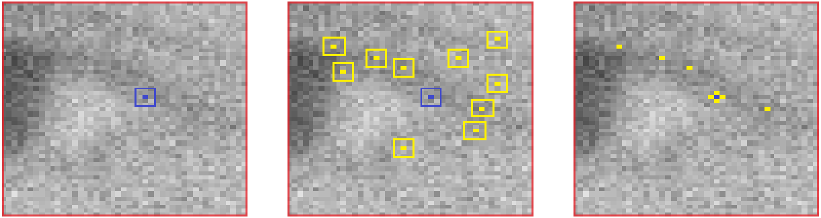
Jeffreys-Kullback

$$\varphi^*(\zeta^*) = W(e^{1-\zeta^*}) + \big(W(e^{1-\zeta^*})\big)^{-1} + \zeta^* - 2$$

Application to image restoration

Non-local Total Variation:

$$\mathrm{NLTV}(x) = \sum_{s \in \mathcal{A}} \left(\sum_{n \in \mathcal{N}_s \subset \mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p}$$



NLTV as dissimilarity measure:

$$\mathrm{NLTV}(x) = \sum_{s \in \mathcal{A}} \left(\sum_{n \in \mathcal{N}_s \subset \mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p}$$

NLTV as dissimilarity measure:

$$\begin{aligned}\mathrm{NLTV}(x) &= \sum_{s\in\mathcal{A}} \left(\sum_{n\in\mathcal{N}_s\subset\mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p} \\ &= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n} \left(x^{(s)} - x^{(n)} \right) \right]_{n\in\mathcal{N}_s} \right\|_p\end{aligned}$$

NLTV as dissimilarity measure:

$$\mathrm{NLTV}(x) = \sum_{s\in\mathcal{A}} \left(\sum_{n\in\mathcal{N}_s\subset\mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p}$$

$$= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n} \left(x^{(s)} - x^{(n)} \right) \right]_{n\in\mathcal{N}_s} \right\|_p$$

$$= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n}\, x^{(s)} \right]_{n\in\mathcal{N}_s} - \left[\omega_{s,n}\, x^{(n)} \right]_{n\in\mathcal{N}_s} \right\|_p$$

NLTV as dissimilarity measure:

$$\begin{aligned}
\mathrm{NLTV}(x) &= \sum_{s\in\mathcal{A}} \left(\sum_{n\in\mathcal{N}_s\subset\mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p} \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n} \left(x^{(s)} - x^{(n)} \right) \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n}\, x^{(s)} \right]_{n\in\mathcal{N}_s} - \left[\omega_{s,n}\, x^{(n)} \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \| A_s x - B_s x \|_p
\end{aligned}$$

NLTV as dissimilarity measure:

$$
\begin{aligned}
\mathrm{NLTV}(x) &= \sum_{s\in\mathcal{A}} \left(\sum_{n\in\mathcal{N}_s\subset\mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p} \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n} \left(x^{(s)} - x^{(n)} \right) \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n}\, x^{(s)} \right]_{n\in\mathcal{N}_s} - \left[\omega_{s,n}\, x^{(n)} \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \| A_s x - B_s x \|_p \\
&= D(Ax, Bx) \qquad \text{(with } A = [A_s]_{s\in\mathcal{A}} \text{ and } B = [B_s]_{s\in\mathcal{A}})
\end{aligned}
$$

NLTV as dissimilarity measure:

$$\begin{aligned}
\mathrm{NLTV}(x) &= \sum_{s\in\mathcal{A}} \left(\sum_{n\in\mathcal{N}_s\subset\mathcal{W}_s} \omega_{s,n} |x^{(s)} - x^{(n)}|^p \right)^{1/p} \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n} \left(x^{(s)} - x^{(n)} \right) \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \left\| \left[\omega_{s,n}\, x^{(s)} \right]_{n\in\mathcal{N}_s} - \left[\omega_{s,n}\, x^{(n)} \right]_{n\in\mathcal{N}_s} \right\|_p \\
&= \sum_{s\in\mathcal{A}} \| A_s x - B_s x \|_p \\
&= D(Ax, Bx) \qquad (\text{with } A = [A_s]_{s\in\mathcal{A}} \text{ and } B = [B_s]_{s\in\mathcal{A}})
\end{aligned}$$

$\Rightarrow$ use more general forms for D.

Dissimilarity based on ℓ_p-norms:

$$D(a,b) = \sum_{s \in \mathcal{A}} \left\| a^{(s)} - b^{(s)} \right\|_p$$

where $a = Ax = (a^{(s)})_{s \in \mathcal{A}}$ and $b = Bx = (b^{(s)})_{s \in \mathcal{A}}$

Dissimilarity based on ℓ_p-norms:

$$D(a,b) = \sum_{s \in \mathcal{A}} \left\| a^{(s)} - b^{(s)} \right\|_p$$

where $a = Ax = (a^{(s)})_{s \in \mathcal{A}}$ *and* $b = Bx = (b^{(s)})_{s \in \mathcal{A}}$

Dissimilarity based on φ-divergences:

$$\boxed{D(a,b) = \sum_{s \in \mathcal{A}} \sum_{m=1}^{|\mathcal{N}_s|} b^{(s,m)} \varphi\left(\frac{a^{(s,m)}}{b^{(s,m)}}\right)}$$

where $a^{(s)} = (a^{(s,m)})_{1 \le m \le |\mathcal{N}_s|}$ *and* $b^{(s)} = (b^{(s,m)})_{1 \le m \le |\mathcal{N}_s|}$

DEGRADATION MODEL

$$\boxed{z = H\overline{w} + n}$$

- $\overline{w}$: original image in $\mathbb{R}^N$,
- H: linear operator from $\mathbb{R}^N$ to $\mathbb{R}^Q$,
- n: zero-mean white Gaussian noise in $\mathbb{R}^Q$,
- z: degraded image of size Q.

DEGRADATION MODEL

$$\boxed{z = H\overline{w} + n}$$

- $\overline{w}$: original image in $\mathbb{R}^N$,
- H: linear operator from $\mathbb{R}^N$ to $\mathbb{R}^Q$,
- n: zero-mean white Gaussian noise in $\mathbb{R}^Q$,
- z: degraded image of size Q.

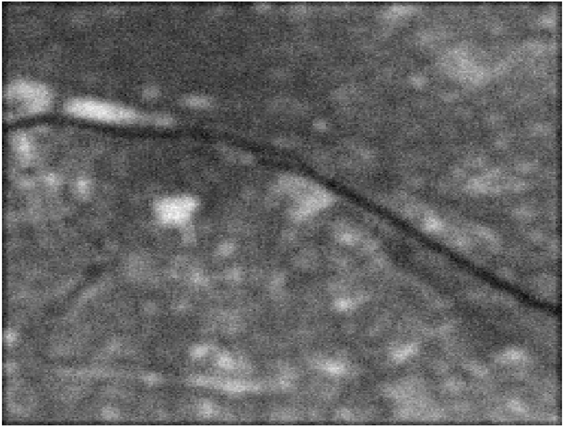
$$\underset{w\in\mathbb{R}^N}{\text{minimize}} \; \underbrace{\frac{1}{2\lambda}\|Hw - z\|^2}_{\text{Data fidelity term}} + \underbrace{D(Aw, Bw)}_{\substack{\text{Regularization term}\\ A,B\in\mathbb{R}^{P\times N}}} + \underbrace{\iota_C(w)}_{\substack{\text{Convex constraint}\\ C=[0,255]^N}}$$

$\lambda \in]0, +\infty[$.

Results

Original image:

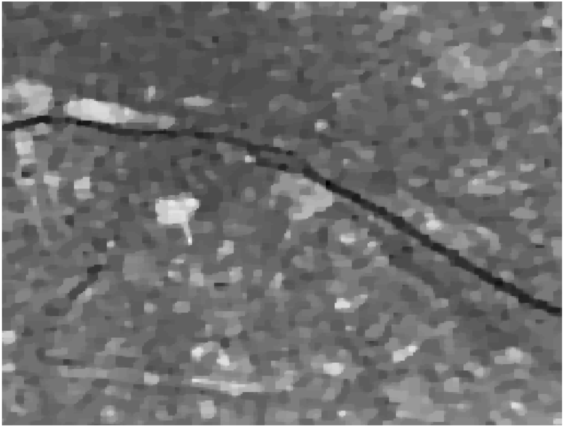
Results



Original image:

Degraded image:
SNR= 13.14 dB, SSIM=0.284

RESULTS

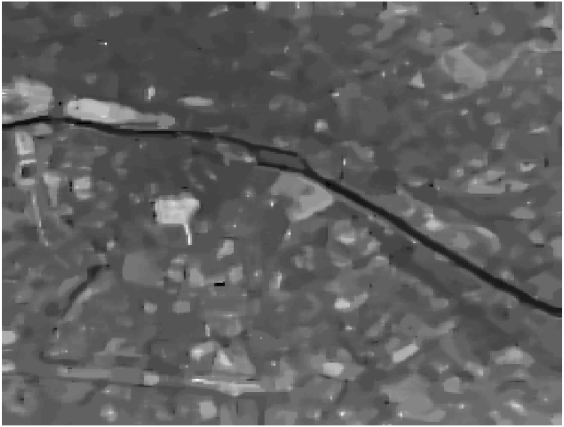


Original image:

Degraded image:
SNR= 13.14 dB, SSIM=0.284

$\ell_{1,2} - \mathrm{TV}$ **result:**
SNR= 15.29 dB, SSIM=0.467

RESULTS



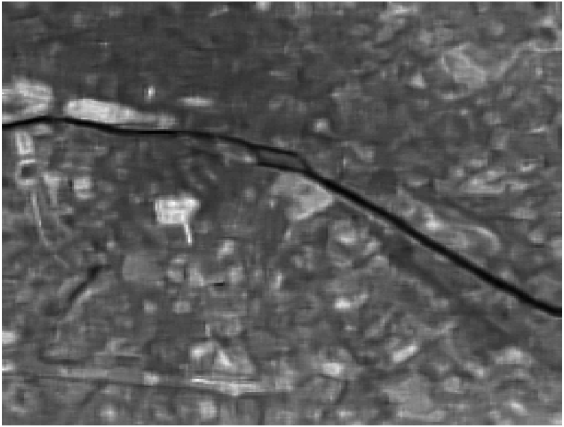
Original image:

Degraded image:
SNR= 13.14 dB, SSIM=0.284

$\ell_{1,2}$ − TV **result:**
SNR= 15.29 dB, SSIM=0.467

$\ell_{1,2}$ − NLTV **result:**
SNR= 15.70 dB, SSIM=0.504

RESULTS



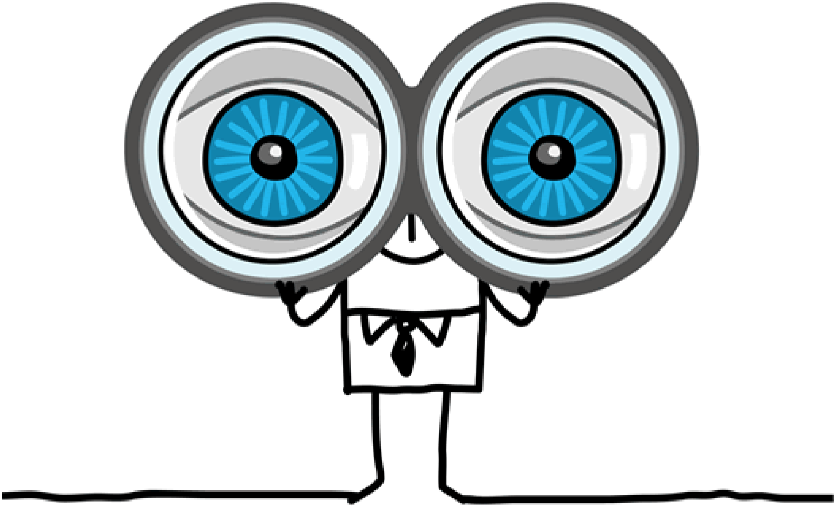
Original image:

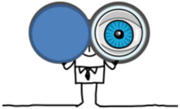
Degraded image:
SNR= 13.14 dB, SSIM=0.284

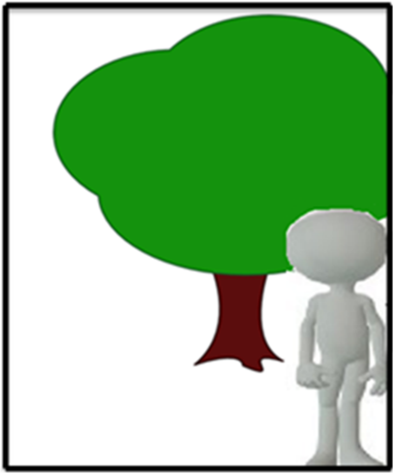
$\ell_{1,2}$ − TV **result:**
SNR= 15.29 dB, SSIM=0.467

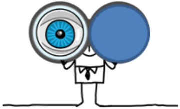
$\ell_{1,2}$ − NLTV **result:**
SNR= 15.70 dB, SSIM=0.504

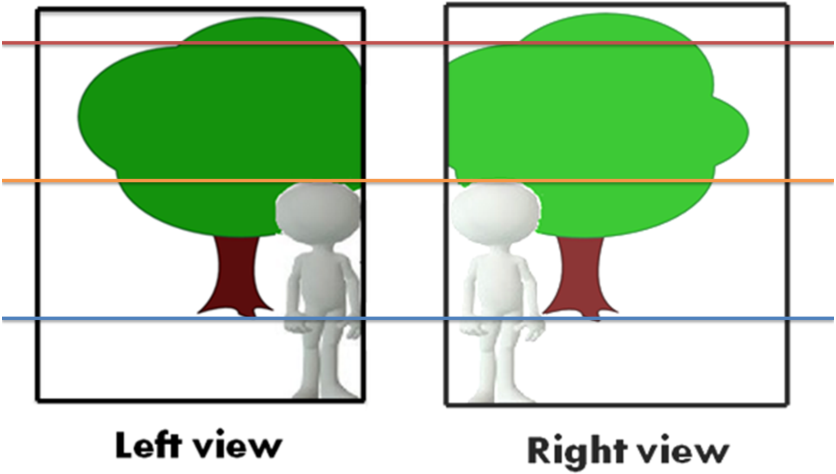
JK − NLTV **result:**
SNR= 16.01 dB, SSIM=0.548

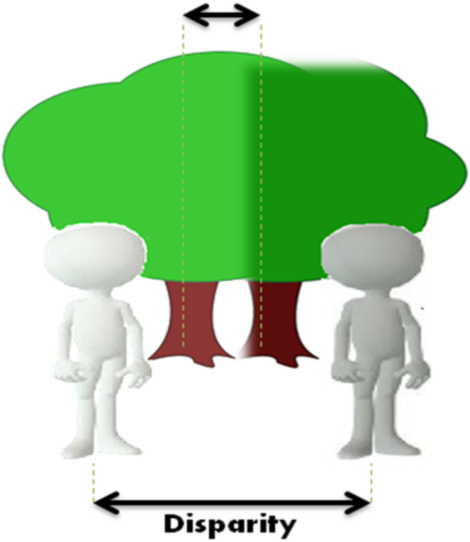






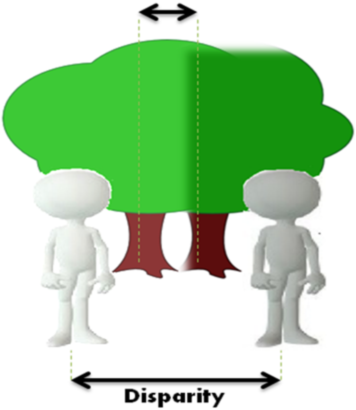



Left view
Right view

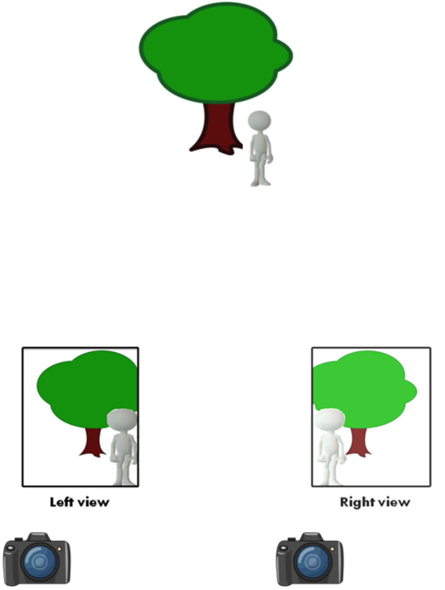

Disparity

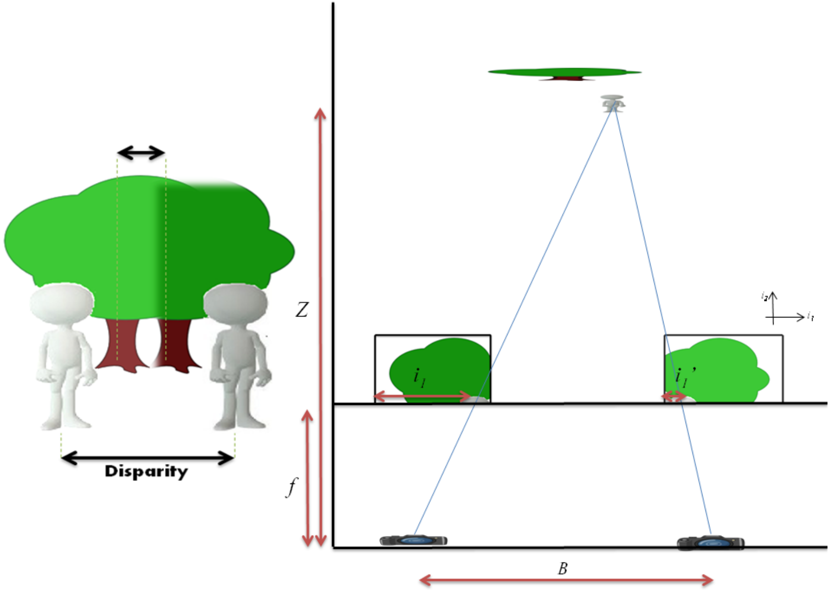

Disparity


Disparity


Disparity


Disparity


Left view
Right view


Disparity
Z
f
B
i_1
i_1'

Disparity

Definition

$$
\begin{aligned}
u : \quad & \mathbb{R}^2 \mapsto \mathbb{R}^2 \\
& (i_1, i_2) \mapsto (i_1 - i'_1, i_2 - i'_2)
\end{aligned}
$$

DISPARITY

Definition

$$\begin{aligned} u: \quad & \mathbb{R}^2 \mapsto \mathbb{R}^2 \\ & (i_1, i_2) \mapsto (i_1 - i_1', i_2 - i_2') \end{aligned}$$

DISPARITY

Definition

$$u: \quad \mathbb{R}^2 \mapsto \mathbb{R}$$
$$(i_1, i_2) \mapsto i_1 - i_1'$$

DISPARITY

Definition

$$u(i_1, i_2) = i_1 - i'_1 = \frac{Bf}{Z}$$

DISPARITY

Definition

$$u(i_1, i_2) = i_1 - i_1' = \frac{Bf}{Z}$$

Objective

Find for each pixel in the left image $I_1 : \mathbb{R}^2 \mapsto \mathbb{R}^K$ a corresponding pixel in the right image $I_2 : \mathbb{R}^2 \mapsto \mathbb{R}^K$.

$$I_1(i_1, i_2) = I_2(i_1', i_2')$$

Disparity

Definition

$$u(i_1, i_2) = i_1 - i_1' = \frac{Bf}{Z}$$

Objective

Find for each pixel in the left image $I_1 : \mathbb{R}^2 \mapsto \mathbb{R}^K$ a corresponding pixel in the right image $I_2 : \mathbb{R}^2 \mapsto \mathbb{R}^K$.

$$I_1(i_1, i_2) = I_2(i_1 - u(i_1, i_2), i_2)$$

DISPARITY

Definition

$$u(i_1, i_2) = i_1 - i_1' = \frac{Bf}{Z}$$

Objective

Find for each pixel in the left image $I_1 : \mathbb{R}^2 \mapsto \mathbb{R}^K$ a corresponding pixel in the right image $I_2 : \mathbb{R}^2 \mapsto \mathbb{R}^K$.

$$v(i_1, i_2) I_1(i_1, i_2) = I_2(i_1 - u(i_1, i_2), i_2)$$

$v : \mathbb{R}^2 \to [0, +\infty[$

PROBLEM FORMULATION

Let $\mathbf{s} = (i_1, i_2)$

Let $\mathcal{A}$ be the image support and $\mathcal{O}$ be the occlusion pixels.

Variational method

$$\widetilde{J}(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{K} \sum_{\mathbf{s} \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}\big(v(\mathbf{s}) I_1^{(k)}(\mathbf{s}) - I_2^{(k)}(i_1 - u(\mathbf{s}), i_2)\big)$$

$\forall k \in \{1, \dots, K\}$, $\phi^{(k)}$ belongs to $\Gamma_0(\mathbb{R})$.

PROBLEM FORMULATION

Let $\mathbf{s} = (i_1, i_2)$

Let $\mathcal{A}$ be the image support and $\mathcal{O}$ be the occlusion pixels.

Variational method

$$\widetilde{J}(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{K} \sum_{\mathbf{s} \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}\big(v(\mathbf{s}) I_1^{(k)}(\mathbf{s}) - I_2^{(k)}(i_1 - u(\mathbf{s}), i_2)\big)$$

$\forall k \in \{1, \ldots, K\}$, $\phi^{(k)}$ belongs to $\Gamma_0(\mathbb{R})$.

$\widetilde{J}$ is non-convex w.r.t. the variable $\mathbf{u}$.

Problem formulation

Variational method

$$\widetilde{J}(\mathbf{u},\mathbf{v}) = \sum_{k=1}^{K} \sum_{\mathbf{s}\in\mathcal{A}\setminus\mathcal{O}} \phi^{(k)}\left(v(\mathbf{s})I_1^{(k)}(\mathbf{s}) - I_2^{(k)}(i_1 - u(\mathbf{s}), i_2)\right)$$

PROBLEM FORMULATION

Variational method

$$\widetilde{J}(\mathbf{u},\mathbf{v}) = \sum_{k=1}^{K} \sum_{\mathbf{s}\in\mathcal{A}\setminus\mathcal{O}} \phi^{(k)}\big(v(\mathbf{s}) I_1^{(k)}(\mathbf{s}) - I_2^{(k)}(i_1 - u(\mathbf{s}), i_2)\big)$$

Convex relaxation:

- First-order Taylor expansion of the disparity compensated right image around an initial value

Problem formulation

Variational method

$$\widetilde{J}(\mathbf{u},\mathbf{v}) = \sum_{k=1}^{K} \sum_{\mathbf{s}\in\mathcal{A}\setminus\mathcal{O}} \phi^{(k)}\big(v(\mathbf{s})I_1^{(k)}(\mathbf{s}) - I_2^{(k)}(i_1 - u(\mathbf{s}), i_2)\big)$$

Convex relaxation:

- First-order Taylor expansion of the disparity compensated right image around an initial value

  for every $k \in \{1,\ldots,K\}$ and $\mathbf{s} \in \mathcal{A}$,

$$\boxed{I_2^{(k)}(i_1 - u(\mathbf{s}), i_2) \simeq I_2^{(k)}(i_1 - \bar{u}(\mathbf{s}), i_2) - (u(\mathbf{s}) - \bar{u}(\mathbf{s}))\nabla^{(1)} I_2^{(k)}(i_1 - \bar{u}(\mathbf{s}), i_2)}$$

where $\nabla^{(1)} I_2^{(k)}$ denotes the horizontal gradient of the k-th component of the right image.

Convex formulation

$$J(\mathbf{u},\mathbf{v}) = \sum_{k=1}^{K} \sum_{s\in\mathcal{A}\setminus\mathcal{O}} \phi^{(k)}\left(T_1^{(k)}(\mathbf{s})u(\mathbf{s}) + T_2^{(k)}(\mathbf{s})v(\mathbf{s}) - r^{(k)}(\mathbf{s})\right)$$

where, for every $k \in \{1,\ldots,K\}$ and $\mathbf{s} \in \mathcal{A}$,

$$\begin{cases} T_1^{(k)}(\mathbf{s}) = \nabla^{(1)} I_2^{(k)}(i_1 - \bar{u}(\mathbf{s}), i_2) \\ T_2^{(k)}(\mathbf{s}) = I_1^{(k)}(\mathbf{s}) \\ r^{(k)}(\mathbf{s}) = I_2^{(k)}(i_1 - \bar{u}(\mathbf{s}), i_2) + \bar{u}(\mathbf{s}) T_1^{(k)}(\mathbf{s}). \end{cases}$$

Let $\mathbf{w} = (\mathbf{u},\mathbf{v})$, $(\forall \mathbf{s} \in \mathcal{A})\ w(\mathbf{s}) = \begin{bmatrix} u(\mathbf{s}) \\ v(\mathbf{s}) \end{bmatrix}$, $\mathbf{T}^{(k)}(\mathbf{s}) = [T_1^{(k)}(\mathbf{s}), T_2^{(k)}(\mathbf{s})]$

$$J(\mathbf{w}) = \sum_{k=1}^{K} \sum_{s\in\mathcal{A}\setminus\mathcal{O}} \phi^{(k)}\left(\mathbf{T}^{(k)}(\mathbf{s})w(\mathbf{s}) - r^{(k)}(\mathbf{s})\right)$$

SET THEORETIC ESTIMATION

$$J(\mathbf{w}) = \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s}) w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

Advantages

- Ability to consider multicomponent images with illumination variation

SET THEORETIC ESTIMATION

$$J(\mathbf{w}) = \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s}) w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

Advantages

- Ability to consider multicomponent images with illumination variation
- Flexibility in minimizing various convex similarity measures (ℓ_1, ℓ_2, divergences …)

SET THEORETIC ESTIMATION

$$J(\mathbf{w}) = \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s})w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

Advantages

- Ability to consider multicomponent images with illumination variation
- Flexibility in minimizing various convex similarity measures (ℓ_1, ℓ_2, divergences …)

Proximity operator ✓

SET THEORETIC ESTIMATION

$$J(\mathbf{w}) = \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s})w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

Advantages

- Ability to consider multicomponent images with illumination variation
- Flexibility in minimizing various convex similarity measures (ℓ_1, ℓ_2, divergences ...)

Proximity operator ✓

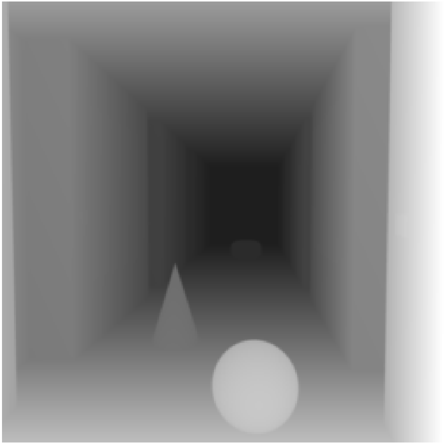
- The minimization of functional J is an ill-posed problem.

  (infinite number of solutions due to the fact that two variables have to be determined for each pixel).
- Additional **constraints** are required to regularize the solution.

Convex Constraints

CONVEX CONSTRAINTS

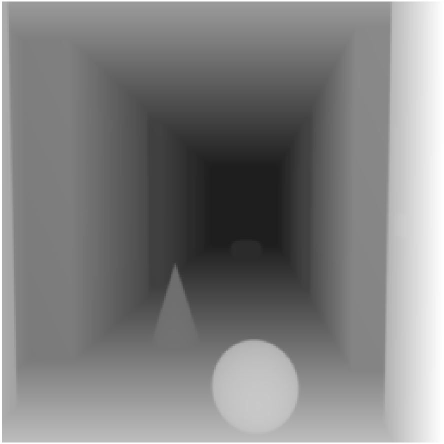
Convex Constraints



Range values:

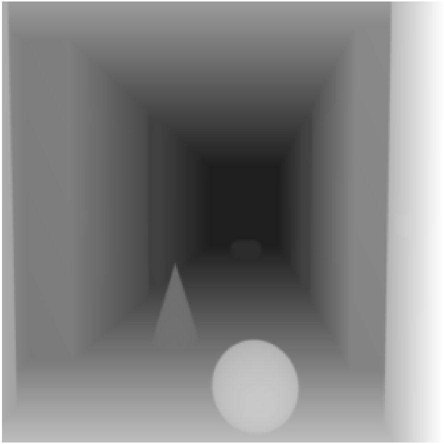
$S_{1,1} = \{\mathbf{u} \in \mathbb{R}^{|\mathcal{A}|} \mid (\forall \mathbf{s} \in \mathcal{A})\ u_{\min} \le u \le u_{\max}\}$, $u_{\min} \ge 0$

CONVEX CONSTRAINTS



Total variation:
$S_{1,2} = \{\mathbf{u} \in \mathbb{R}^{|\mathcal{A}|} \mid \mathrm{TV}(\mathbf{u}) \leq \tau_2\}$,
$\tau_2 \geq 0$
$\mathrm{TV}(\mathbf{u}) = \sum_{\mathbf{s}\in\mathcal{A}} \sqrt{|\widehat{\nabla^{(1)}} u(\mathbf{s})|^2 + |\widehat{\nabla^{(2)}} u(\mathbf{s})|^2}$
$\widehat{\nabla^{(1)}}$ and $\widehat{\nabla^{(2)}}$: discrete gradients

Convex Constraints

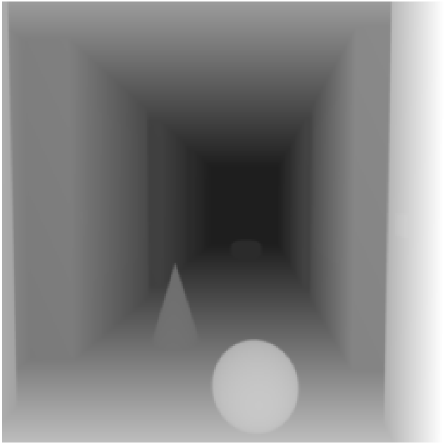


Frame analysis constraint:

$S'_{1,2} =$
$\{\mathbf{u} \in \mathbb{R}^{|\mathcal{A}|} \mid \sum_{q=1}^{Q} \eta_q |(F\mathbf{u})_q| \leq \tau'_2\}$

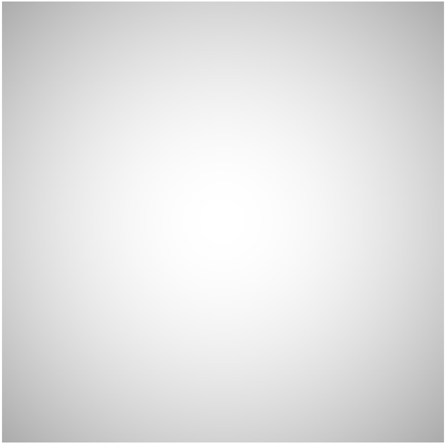
$F : \mathbb{R}^{|\mathcal{A}|} \to \mathbb{R}^Q$ with $Q \geq |\mathcal{A}|$,
$(\eta_q)_{1 \leq q \leq Q} \in [0, +\infty[^Q$ and $\tau'_2 > 0$.
$F^\top F = \nu I$, where $\nu > 0$

CONVEX CONSTRAINTS



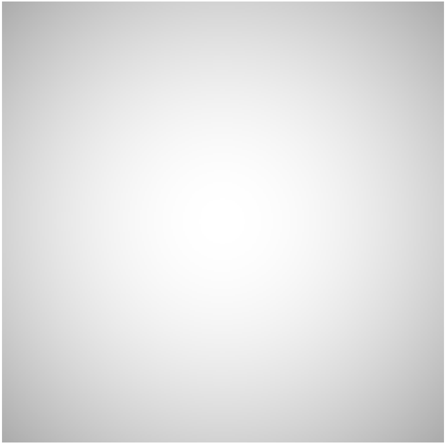
Second-order constraint:
$S_{1,3} = \left\{ \mathbf{u} \in \mathbb{R}^{|\mathcal{A}|} \mid \mathrm{TV}_2(\mathbf{u}) \leq \tau_3 \right\}$,
$\tau_3 > 0$.
$\mathrm{TV}_2(\mathbf{u}) = \sum_{\mathbf{s}\in\mathcal{A}} \sqrt{|\widehat{\nabla}^2 \mathbf{u}(\mathbf{s}) u(\mathbf{s})|^2}$
$\widehat{\nabla}^2 \mathbf{u}(\mathbf{s})$: discrete Hessian operator

Convex Constraints



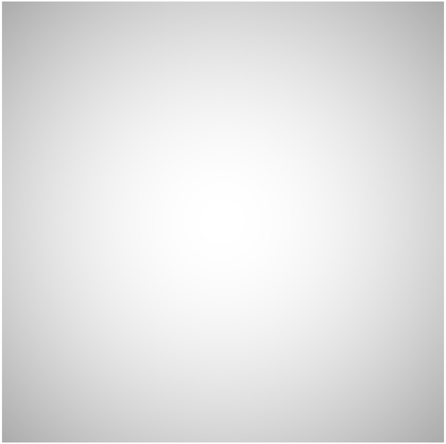
Range values:
$S_{2,1} = \{\mathbf{v} \in \mathbb{R}^{|\mathcal{A}|} \mid (\forall \mathbf{s} \in \mathcal{A})\ v_{\min} \leq v(\mathbf{s}) \leq v_{\max}\}$, $v_{\min} \geq 0$

Convex Constraints



First-order smoothness constraint:
$S_{2,2} = \{\mathbf{v} \in \mathbb{R}^{|\mathcal{A}|} \mid \|\widehat{\nabla}\mathbf{v}\|_{\ell_2}^2 \leq \kappa_2\}$,
$\kappa_2 > 0$
$\|\widehat{\nabla}\mathbf{v}\|_{\ell^2} =$
$(\sum_{\mathbf{s}\in\mathcal{A}} |\widehat{\nabla^{(1)}} v(\mathbf{s})|^2 + |\widehat{\nabla^{(2)}} v(\mathbf{s})|^2)^{1/2}$.

CONVEX CONSTRAINTS



Second-order constraint:
$S_{2,3} = \left\{ \mathbf{v} \in \mathbb{R}^{|\mathcal{A}|} \;\middle|\; \|\widehat{\nabla^2}\mathbf{v}\|_{\ell_2}^2 \le \kappa_3 \right\},$
$\kappa_3 > 0$
$\widehat{\nabla^2}\mathbf{u}(\mathbf{s})$: discrete Hessian operator

Proposed approach

General formulation

$$\underset{L_i w \in C_i, i \in \{1,\dots,m\}}{\text{minimize}} \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s}) w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

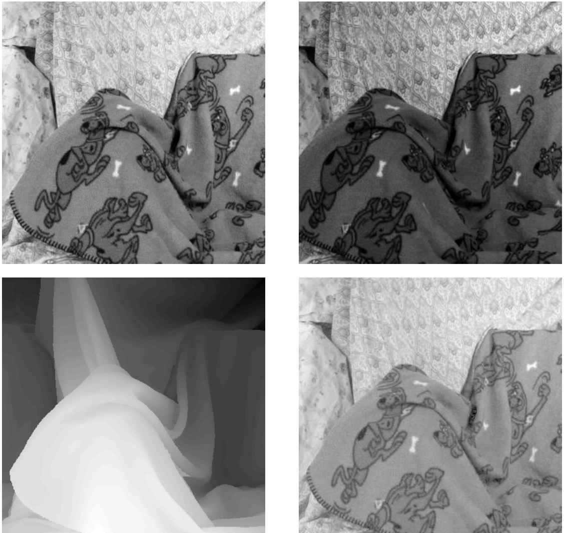
PROPOSED APPROACH

General formulation

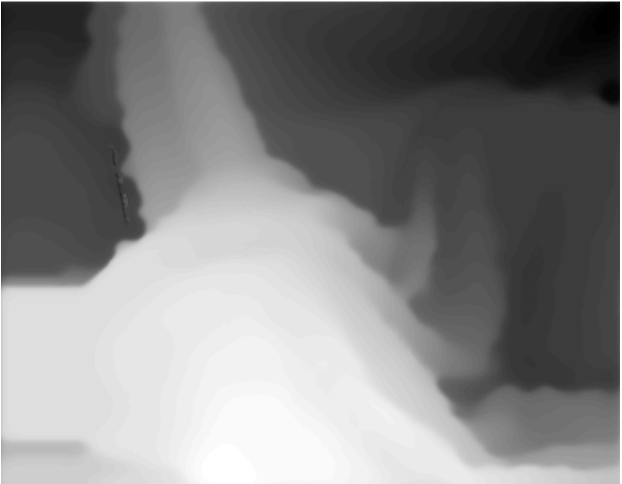
$$\underset{L_i w \in C_i, i \in \{1,\ldots,m\}}{\text{minimize}} \sum_{k=1}^{K} \sum_{s \in \mathcal{A} \setminus \mathcal{O}} \phi^{(k)}(\mathbf{T}^{(k)}(\mathbf{s}) w(\mathbf{s}) - r^{(k)}(\mathbf{s}))$$

- The PPXA+ algorithm can be employed to minimize J on some closed convex constraint sets $(C_i)_{1 \leq i \leq m}$.
- It consists of computing, in parallel, the projections onto the different convex sets and the proximity operator of the criterion J.

Results (gray level images)

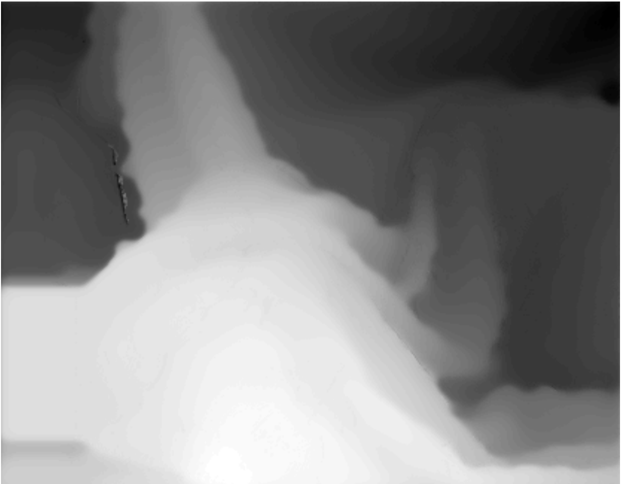


RESULTS (GRAY LEVEL IMAGES)



ℓ_2-norm:
MAE= 0.83, Err = 3.62%

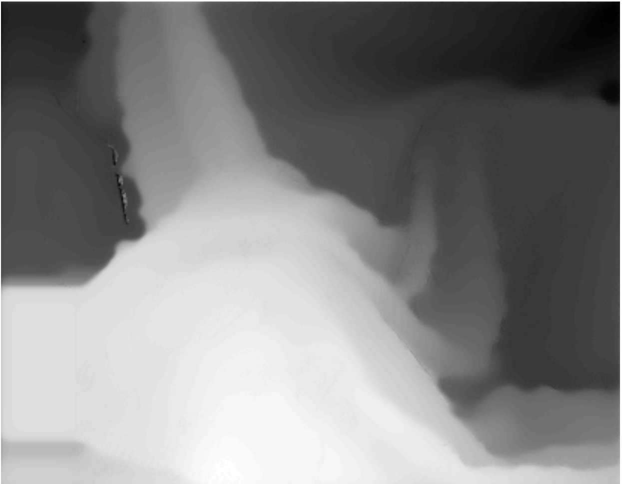
RESULTS (GRAY LEVEL IMAGES)



ℓ_2-norm:
MAE= 0.83, Err = 3.62%

Kullback-Leibler:
MAE= 0.82, Err = 3.36%

Results (gray level images)



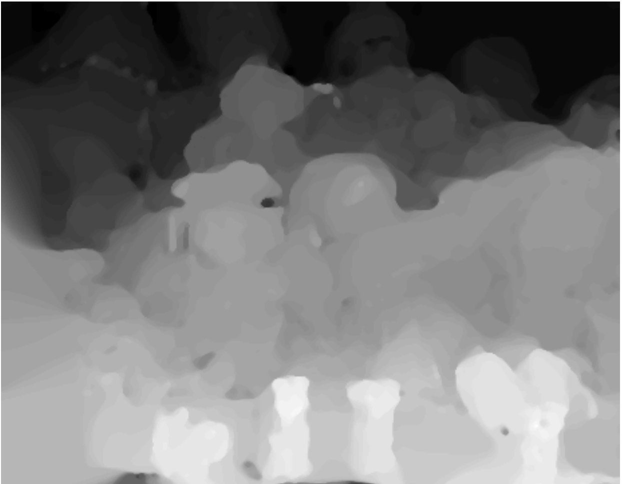
ℓ_2-norm:
MAE= 0.83, Err = 3.62%

Kullback-Leibler:
MAE= 0.82, Err = 3.36%

Jeffreys-Kullback:
MAE= 0.83, Err= 3.44%

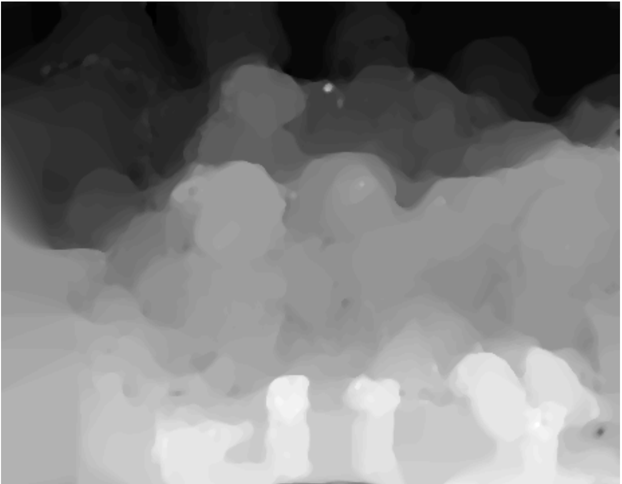
Results (ℓ_1-norm)

Results (ℓ_1-norm)



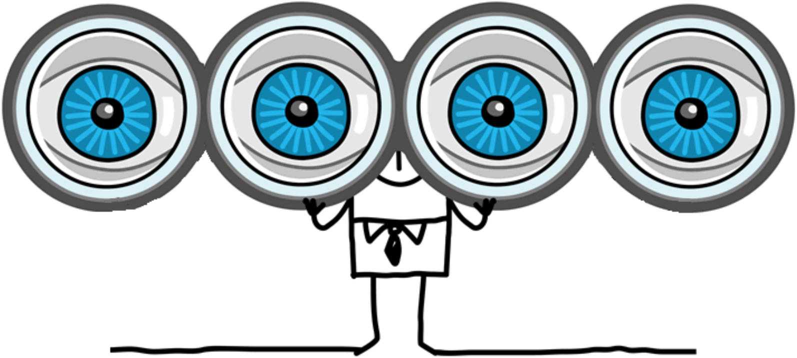
Gray level images:
MAE= 1.26, Err = 13%

RESULTS (ℓ_1-NORM)



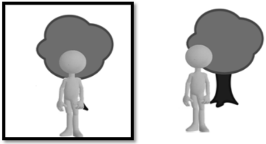
Gray level images:
MAE= 1.26, Err = 13%

color images:
MAE= 1.10, Err = 11%



Multi-label approach
Multiple images

DISPARITY ESTIMATION



Variational method

$$f(\mathbf{u}) = \sum_{\mathbf{s}\in\mathcal{A}} \psi(I_1(\mathbf{s}), I_2(i_1 - u(\mathbf{s}), i_2))$$

$\psi \in \Gamma_0(\mathbb{R})$.

Multi-label approach

The disparity u is quantized over $Q+1$ quantization levels $r_0, r_1, \ldots, r_Q$ $(r_0 < r_1 < \cdots < r_Q)$

$$(\forall\, \mathbf{s} \in \mathcal{A}) \qquad u(\mathbf{s}) = r_0 + \sum_{q=1}^{Q} (r_q - r_{q-1})\theta_q(\mathbf{s})$$

where $\theta = (\theta_1, \ldots, \theta_Q) \in \mathcal{B}$ such that

$$(\forall\, q \in \{1, \ldots, Q\})(\forall\, \mathbf{s} \in \mathcal{A}) \qquad \theta_q(\mathbf{s}) = \begin{cases} 1 & \text{if } \; u(\mathbf{s}) \geq r_q \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{B} = \{\theta \in (\{0,1\}^P)^Q \,|\, (\forall\, \mathbf{s} \in \mathcal{A}) \;\; 1 \geq \theta_1(\mathbf{s}) \geq \cdots \geq \theta_Q(\mathbf{s}) \geq 0\}.$$

CONVEX FORMULATION CREMERS ET AL. [2011]

$$\widetilde{f}(\theta) = \sum_{\mathbf{s}\in\mathcal{A}} \sum_{q=0}^{Q} \psi(I_1(\mathbf{s}), I_2(i_1 - r_q, i_2))(\theta_q(\mathbf{s}) - \theta_{q+1}(\mathbf{s}))$$

The minimization problem can be expressed as:

$$\underset{\theta\in\mathcal{B}}{\text{minimize}} \;\; \widetilde{f}(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\,\mathsf{tv}(\theta_q), \quad \mu > 0.$$

Convex formulation Cremers et al. [2011]

$$\widetilde{f}(\theta) = \sum_{\mathbf{s}\in\mathcal{A}} \sum_{q=0}^{Q} \psi(I_1(\mathbf{s}), I_2(i_1 - r_q, i_2))(\theta_q(\mathbf{s}) - \theta_{q+1}(\mathbf{s}))$$

The minimization problem can be expressed as:

$$\underset{\theta\in\mathcal{B}}{\text{minimize}} \;\; \widetilde{f}(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_q), \qquad \mu > 0.$$

where

$$\mathcal{B} = \{\theta \in (\{0,1\}^P)^Q \,|\, (\forall\, \mathbf{s} \in \mathcal{A}) \;\; 1 \geq \theta_1(\mathbf{s}) \geq \cdots \geq \theta_Q(\mathbf{s}) \geq 0\}.$$

Convex formulation — Cremers et al. [2011]

$$\widetilde{f}(\theta) = \sum_{\mathbf{s}\in\mathcal{A}} \sum_{q=0}^{Q} \psi(I_1(\mathbf{s}), I_2(i_1 - r_q, i_2))(\theta_q(\mathbf{s}) - \theta_{q+1}(\mathbf{s}))$$

The minimization problem can be expressed as:

$$\underset{\theta\in\mathcal{B}}{\text{minimize}} \;\; \widetilde{f}(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_q), \quad \mu > 0.$$

Convex relaxation:

$$\underset{\theta\in\mathcal{R}}{\text{minimize}} \;\; \widetilde{f}(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_q)$$

where

$$\mathcal{R} = \{\theta \in ([0,1]^P)^Q \,|\, (\forall\, \mathbf{s} \in \mathcal{A}) \;\; 1 \geq \theta_1(\mathbf{s}) \geq \cdots \geq \theta_Q(\mathbf{s}) \geq 0\}.$$

Optimization problem

$$\underset{\theta\in\mathcal{R}}{\text{minimize}}\ g(\theta)+\mu\sum_{q=1}^{Q}(r_q-r_{q-1})\,\mathrm{tv}(\theta_q)$$

Data fidelity: $g:\mathcal{R}\to\mathbb{R}$, $\psi^j_{1,2}(s)=\psi(I_1(\mathbf{s}),I_2(i_1-j,i_2))$

$$(\forall\,\theta\in\mathcal{R})\qquad g(\theta)=\widetilde{f}(\theta)-\sum_{\mathbf{s}\in\mathcal{A}\setminus\mathcal{O}}\psi^{r_0}_{1,2}(s)=\langle\varsigma\mid\theta\rangle,$$

where $\varsigma=(\varsigma_1,\ldots,\varsigma_Q)\in(\mathbb{R}^{|\mathcal{A}|})^Q$, such that

$$\varsigma_q(\mathbf{s})=1(\mathbf{s})(\psi^{r_q}_{1,2}(s)-\psi^{r_{q-1}}_{1,2}(s))$$

$1(\mathrm{s})=1$ if $\mathrm{s}\in\mathcal{A}_n\setminus\mathcal{O}$ and 0 otherwise.

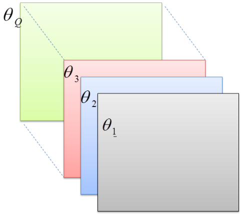
Possibility to handle nonconvex similarity measures

$\psi=|\cdot|$, $\psi=\min\{|\cdot|,\epsilon\}$, $\psi=|\cdot|^{\frac{1}{2}}$, $\psi=\min\{|\cdot|^{\frac{1}{2}},\epsilon\}$

OPTIMIZATION PROBLEM

$$\underset{\theta \in \mathcal{R}}{\text{minimize}} \;\; g(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_q)$$

Regularization Discrete total variation



Proximity operator ✓

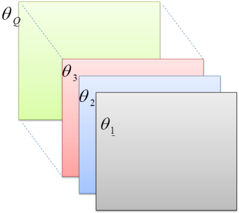
OPTIMIZATION PROBLEM

$$\underset{\theta \in \mathcal{R}}{\text{minimize}} \quad g(\theta) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_q)$$

Convex set

$$\theta_n \in \mathcal{R} \Leftrightarrow (\theta \in E_1 \text{ and } L\theta \in E_2)$$

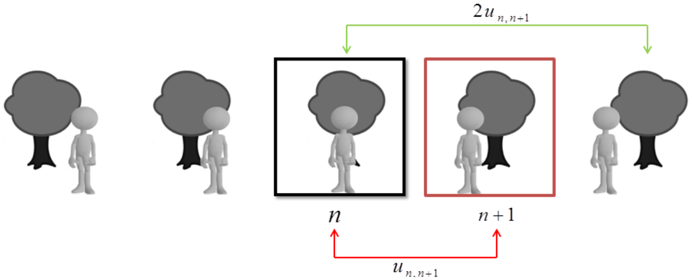
where $E_1 = ([0,1]^P)^Q$, $E_2 = ([0,+\infty[^P)^{Q-1}$ and $L : (\mathbb{R}^P)^Q \to (\mathbb{R}^P)^{Q-1}$ is a linear operator, calculating the successive differences between the Q components of θ



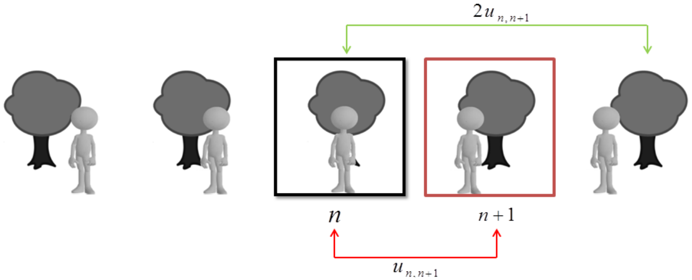
Projection onto closed convex sets E_1 and E_2 ✓

MULTIVIEW DISPARITY ESTIMATION

MULTIVIEW DISPARITY ESTIMATION



Multiview disparity estimation

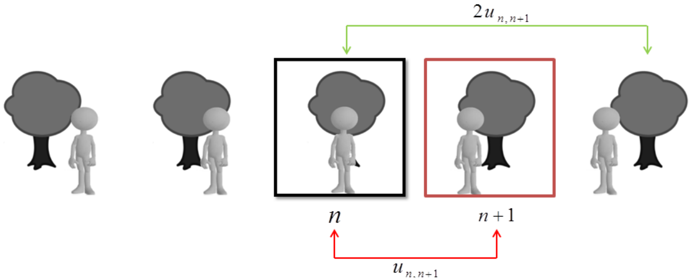


N images

$$(\forall (n,m) \in \{1,\dots,N\}^2, n \neq m) \qquad u_{n,m} = \alpha_{n,m} u_{n,k_n}$$

$\alpha_{n,m} = \frac{m-n}{k_n-n}$

MULTIVIEW DISPARITY ESTIMATION

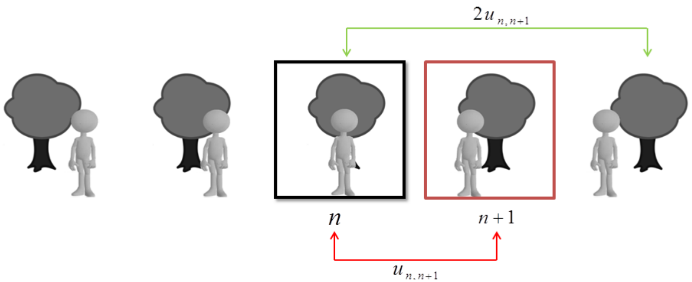


N images

$$\widetilde{f_n}(\mathbf{u}_{n,k_n}) = \sum_{\substack{m=1\\m\neq n}}^{N} \sum_{\mathbf{s}\in\mathcal{D}_{n,m}} \psi(I_n(\mathbf{s}) - I_{k_n}(i_1 - \alpha_{n,m} u_{n,k_n}(\mathbf{s}), i_2))$$

$k_n = n+1$, $\psi \in \Gamma_0(\mathbb{R})$, and $\mathcal{D}_{n,m} \subset \mathcal{A}_n$: unoccluded pixel between n-th and m-th view.

Multiview disparity estimation



N images

$$\widetilde{f}_n(\mathbf{u}_{n,k_n}) = \sum_{\substack{m=1\\m\neq n}}^{N} \sum_{\mathbf{s}\in\mathcal{D}_{n,m}} \psi(I_n(\mathbf{s}) - I_m(i_1 - \alpha_{n,m} u_{n,k_n}(\mathbf{s}), i_2))$$

$k_n = n+1$, $\psi \in \Gamma_0(\mathbb{R})$, and $\mathcal{D}_{n,m} \subset \mathcal{A}_n$: unoccluded pixel between n-th and m-th view.

MULTIVIEW DISPARITY ESTIMATION

Discretization

$$(\forall\, \mathbf{s} \in \mathcal{A}_n) \qquad u_{n,k_n}^{(\mathbf{s})} = r_0 + \sum_{q=1}^{Q} (r_q - r_{q-1}) \theta_{n,q}^{(\mathbf{s})}$$

where

$$(\forall\, q \in \{1,\ldots,Q\})(\forall\, \mathbf{s} \in \mathcal{A}_n) \qquad \theta_{n,q}^{(\mathbf{s})} = \begin{cases} 1 & \text{if } u_{n,k_n}^{(\mathbf{s})} \geq r_q \\ 0 & \text{otherwise} \end{cases}$$

Multiview disparity estimation

$$\underset{\theta_n \in \mathcal{R}}{\text{minimize}} \;\; g_n(\theta_n) + \mu \sum_{q=1}^{Q} (r_q - r_{q-1})\, \mathrm{tv}(\theta_{n,q})$$

Data fidelity: $g_n : \mathcal{R} \to \mathbb{R}$

$$(\forall\, \theta_n \in \mathcal{R}) \qquad g_n(\theta_n) = \sum_{q=1}^{Q} \sum_{\mathbf{s} \in \mathcal{A}_n} \langle \varsigma_{n,q} \mid \theta_{n,q} \rangle = \langle \varsigma_n \mid \theta_n \rangle,$$

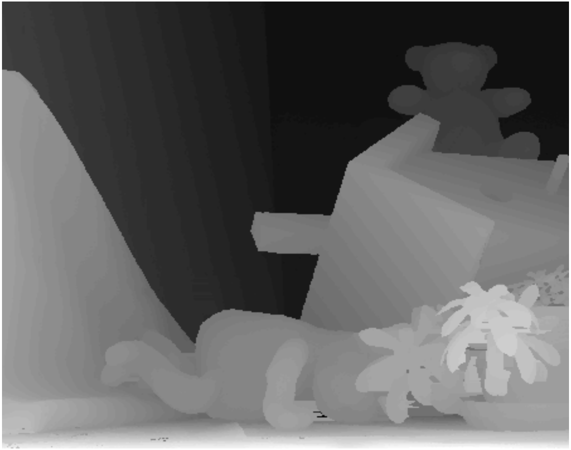
where $\varsigma_n = (\varsigma_{n,1}, \ldots, \varsigma_{n,Q}) \in (\mathbb{R}^P)^Q$, such that

$$\varsigma_{n,q}^{(\mathbf{s})} = \sum_{\substack{m=1 \\ m \neq n}}^{N} 1_{n,m}(\mathbf{s}) (\psi_{n,m}^{\alpha_{n,m} r_q} - \psi_{n,m}^{\alpha_{n,m} r_{q-1}})$$

$1_{n,m}(\mathbf{s}) = 1$ if $\mathbf{s} \in \mathcal{D}_{n,m}$ and 0 otherwise.

RESULTS (TRUNCATED ℓ_1-NORM)

RESULTS (TRUNCATED ℓ_1-NORM)



RESULTS (TRUNCATED ℓ_1-NORM)

Two images:
MAE= 0.56, Err = 4.29%

RESULTS (TRUNCATED ℓ_1-NORM)

Two images:
MAE= 0.56, Err = 4.29%

Three images:
MAE=0.48, Err = 4.08%

RESULTS (TRUNCATED ℓ_1-NORM)

Two images:
MAE= 0.56, Err = 4.29%

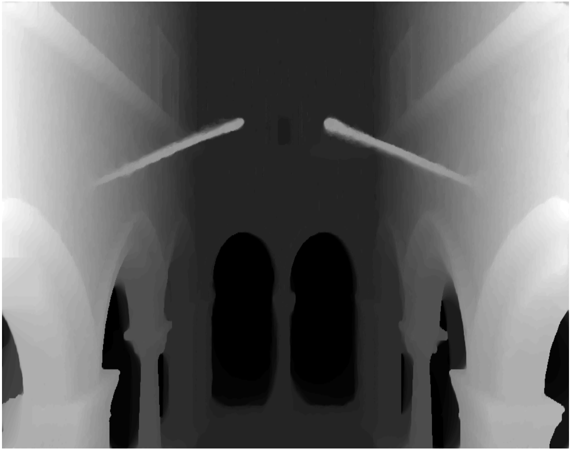
Three images:
MAE=0.48, Err = 4.08%

Five images:
MAE=0.48, Err= 3.82%

Results

RESULTS

RESULTS



Our approach:
Err = 3.10%

RESULTS

Our approach:
Err = 3.10%

Graph cut Woodford et al. [2009]**:**
Err = 4.84%

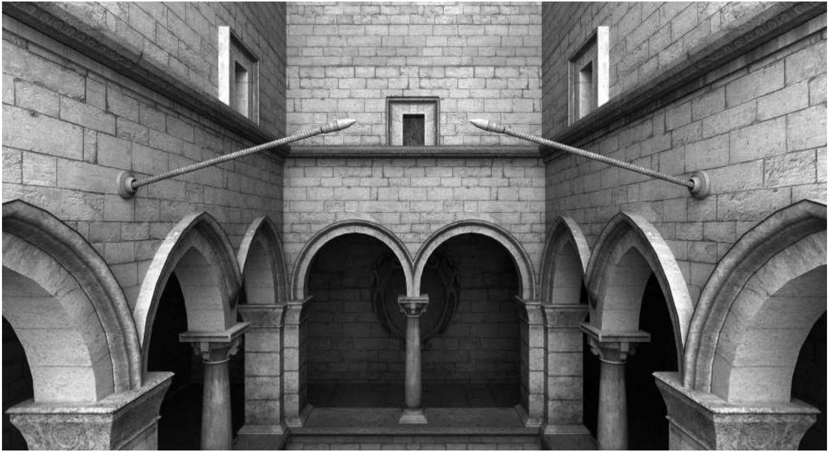
Execution time

$\frac{T_{\mathrm{MultiLabel}}}{T_{\mathrm{Graph-cut}}} = 0.906$

DISPARITY MAP SEQUENCE

Disparity map sequence

Disparity map sequence



DISPARITY MAP SEQUENCE

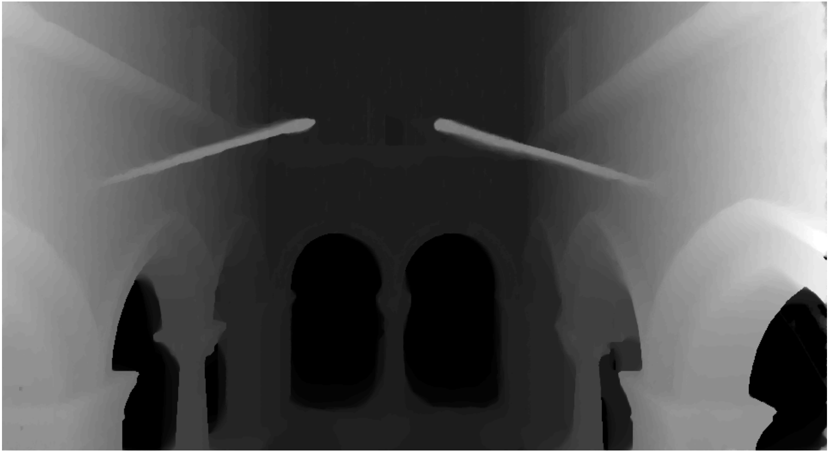
Disparity map sequence

Disparity map sequence

DISPARITY MAP SEQUENCE

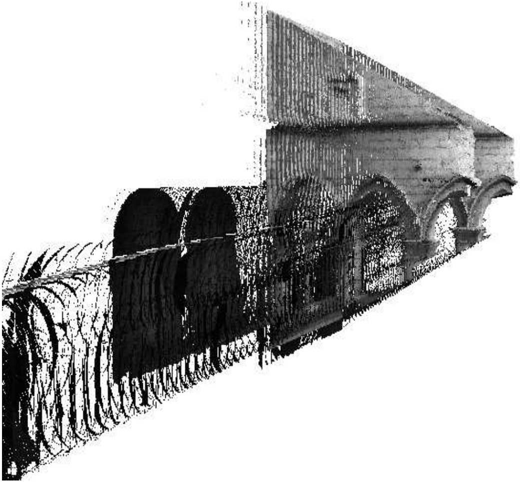
DISPARITY MAP SEQUENCE

Disparity map sequence

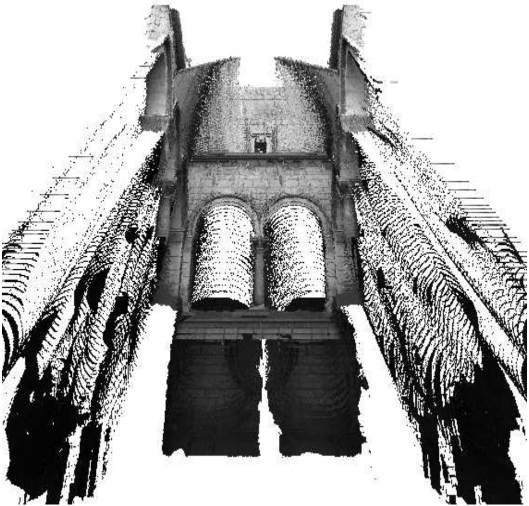


Disparity map sequence

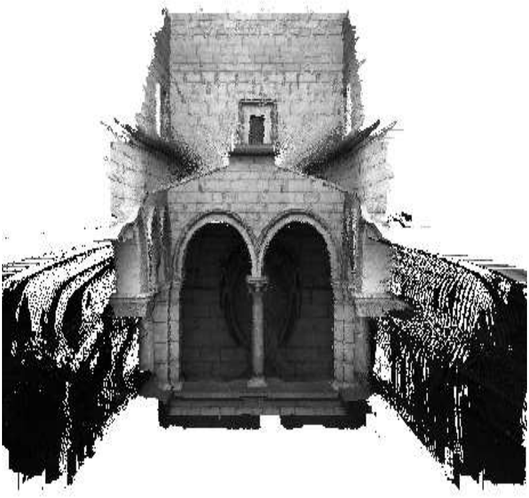
DISPARITY MAP SEQUENCE



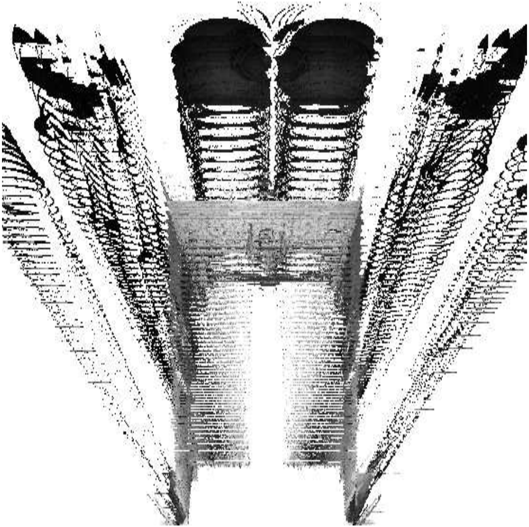
DISPARITY MAP SEQUENCE



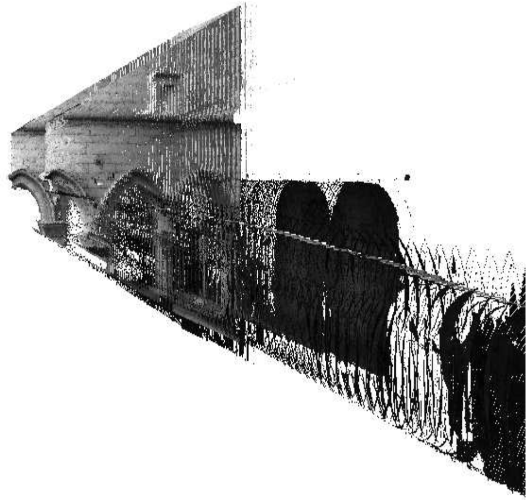
Disparity map sequence



Disparity map sequence



DISPARITY MAP SEQUENCE



CONTRIBUTIONS

Divergence

- New expressions for the proximity operator of several φ-divergences.
- General form of optimization problem (joint minimization w.r.t. of the two variables).
- Application to image restoration.
- Divergence proximity operator for epigraphical projections

CONTRIBUTIONS

Stereo vision

- Evaluation of the potential of a convex optimization approach to deal with disparity estimation under illumination variation.
- Relaxation using Taylor approximation.
- Ability to consider various distance measure and multicomponent images with illumination variation.

CONTRIBUTIONS

Stereo vision

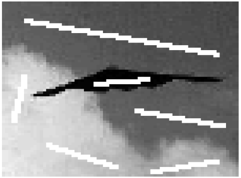
- Evaluation of the potential of a convex optimization approach to deal with disparity estimation under illumination variation.
- Relaxation using Taylor approximation.
- Ability to consider various distance measure and multicomponent images with illumination variation.

Multi-view

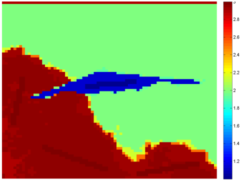
- Convex optimization for disparity map sequence.
- Relaxation based on multilabel approach.
- Possibility of handling nonconvex similarity measures.

Perspectives

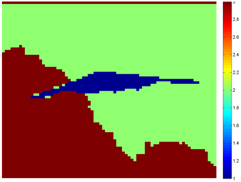
- φ-divergence in segmentation (Histograms based method).



Input image



result



rounded result

PERSPECTIVES

- φ-divergence in blind deconvolution.

PERSPECTIVES

- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.

PERSPECTIVES

- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.
- Exploiting the dependence among the disparity sequence maps.

PERSPECTIVES

- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.
- Exploiting the dependence among the disparity sequence maps.
- Disparity and motion from a multi-view video sequence.

PERSPECTIVES

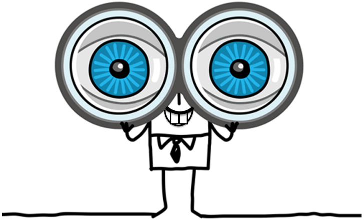
- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.
- Exploiting the dependence among the disparity sequence maps.
- Disparity and motion from a multi-view video sequence.
- Combining the discrete and continuous methods.

PERSPECTIVES

- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.
- Exploiting the dependence among the disparity sequence maps.
- Disparity and motion from a multi-view video sequence.
- Combining the discrete and continuous methods.
- Extension to view synthesis application.

PERSPECTIVES

- φ-divergence in blind deconvolution.
- Epigraphical projection in allocation problem.
- Exploiting the dependence among the disparity sequence maps.
- Disparity and motion from a multi-view video sequence.
- Combining the discrete and continuous methods.
- Extension to view synthesis application.



Thank you

Illumination variation

Artifical illumination

Find for each pixel in the left image $I_1 : \mathbb{R}^2 \mapsto \mathbb{R}^K$ a corresponding pixel in the right image $I_2 : \mathbb{R}^2 \mapsto \mathbb{R}^K$.

$$\forall k \in \{1, \ldots, K\}, \quad v^{(k)}(i_1, i_2) I_1^{(k)}(i_1, i_2) = I_2^{(k)}(i_1 - u(i_1, i_2), i_2)$$

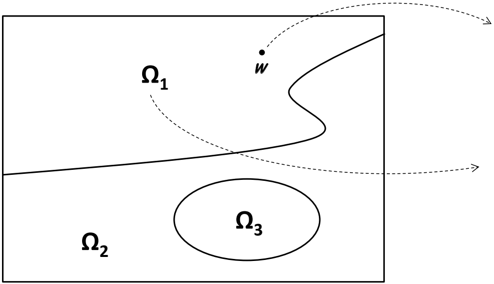
$v : \mathbb{R}^2 \to [0, +\infty[$

- The spectrum of the illumination source changes in function of the power.
- Color changes.
- Illumination variation variable per color component.

Image segmentation

CONSIDERED PROBLEM

Partition the image domain $\Omega = \{1, \ldots, N\}$ into J regions...



Multi-class representation:

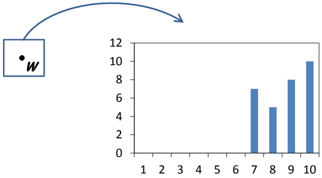
$$u^{(w)} = (u_1^{(w)}, \ldots, u_J^{(w)})$$

Region histogram:

$$p_j = (p_j^{(1)}, \ldots, p_j^{(L)})$$

Considered problem

... so that the local histograms in each region Ω_j are *similar*.



Local histogram:

$$\mathsf{q}_w = (\mathsf{q}_w^{(1)}, \ldots, \mathsf{q}_w^{(L)})$$

VARIATIONAL APPROACH

Multi-label relaxation within a jointly minimization [Qiao et al. 2014]

$$\underset{u,p}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J} \sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} \quad \text{subj. to}$$

$$\begin{cases} (\forall w \in \{1,\ldots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\ldots,J\}) & p_j \quad \in [0,+\infty[^L, \quad \sum_{\ell=1}^{L} p_j^{(\ell)} = 1, \end{cases}$$

where $\lambda > 0$.

Proposed reformulation

- We rewrite the non-convex function as:

$$\sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)}$$

Proposed reformulation

- We rewrite the non-convex function as:

$$\sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} = \sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, p_j^{(\ell)}\, u_j^{(w)})$$

Proposed reformulation

- We rewrite the non-convex function as:

$$\sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} = \sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, p_j^{(\ell)}\, u_j^{(w)})$$

- and we introduce the rank-one matrix:

$$(v_j^{(\ell,w)})_{1\le\ell\le L, 1\le w\le N} = (p_j^{(\ell)}\, u_j^{(w)})_{1\le\ell\le L, 1\le w\le N}$$

Proposed reformulation

- We rewrite the non-convex function as:

$$\sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} = \sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, p_j^{(\ell)}\, u_j^{(w)})$$

- and we introduce the rank-one matrix:

$$v_j = p_j u_j^\top \in \mathbb{R}^{L\times N}$$

PROPOSED REFORMULATION

- We rewrite the non-convex function as:

$$\sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} = \sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, p_j^{(\ell)}\, u_j^{(w)})$$

- and we introduce the rank-one matrix:

$$v_j = p_j u_j^\top \in \mathbb{R}^{L\times N}$$

- so that the above function can be replaced by:

$$\boxed{\sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} \quad \to \quad \sum_{w=1}^{N}\sum_{\ell=1}^{L}\Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, v_j^{(\ell,w)})}$$

Problem reformulation

We reformulate the original problem

$$\underset{u,p}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J} \sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}, p_j^{(\ell)})\, u_j^{(w)} \quad \text{subj. to}$$

$$\begin{cases} (\forall w \in \{1,\ldots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \displaystyle\sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\ldots,J\}) & p_j \quad \in [0,+\infty[^L, \quad \displaystyle\sum_{\ell=1}^{L} p_j^{(\ell)} = 1. \end{cases}$$

PROBLEM REFORMULATION

We reformulate the original problem as follows:

$$\underset{u,v}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J} \sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, v_j^{(\ell,w)}) \quad \text{subj. to}$$

$$\begin{cases} (\forall w \in \{1,\ldots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\ldots,J\}) & v_j \in [0,+\infty[^{L\times N}, \quad v_j = p_j u_j^\top. \end{cases}$$

PROBLEM REFORMULATION

We reformulate the original problem as follows:

$$\underset{u,v}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J} \sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, v_j^{(\ell,w)}) \quad \text{subj. to}$$

$$\begin{cases} (\forall w \in \{1,\ldots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\ldots,J\}) & v_j \in [0,+\infty[^{L\times N}, \quad \sum_{\ell=1}^{L} v_j^{(\ell,w)} = u_j^{(w)}, \\ (\forall j \in \{1,\ldots,J\}) & \operatorname{rank}(v_j) \le 1. \end{cases}$$

CONVEX RELAXATION

We relax the rank-one constraint by the nuclear norm:

$$\underset{u,v}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J} \sum_{w=1}^{N} \sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, v_j^{(\ell,w)}) + \mu \sum_{j=1}^{J} \|v_j\|_*$$

$$\text{subj. to} \quad \begin{cases} (\forall w \in \{1,\dots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \displaystyle\sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\dots,J\}) & v_j \in [0,+\infty[^{L\times N}, \quad \displaystyle\sum_{\ell=1}^{L} v_j^{(\ell,w)} = u_j^{(w)}. \end{cases}$$

INTERACTIVE SEGMENTATION

We also add a constraint to allow for user-defined scribbles:

$$\underset{u,v}{\text{minimize}} \quad \sum_{j=1}^{J} \|\nabla u_j\|_{1,2} + \lambda \sum_{j=1}^{J}\sum_{w=1}^{N}\sum_{\ell=1}^{L} \Phi(\mathsf{q}_w^{(\ell)}\, u_j^{(w)}, v_j^{(\ell,w)}) + \mu \sum_{j=1}^{J} \|v_j\|_*$$

$$\text{subj. to} \quad \begin{cases} (\forall w \in \{1,\ldots,N\}) & u^{(w)} \in [0,+\infty[^J, \quad \sum_{j=1}^{J} u_j^{(w)} = 1, \\ (\forall j \in \{1,\ldots,J\}) & v_j \in [0,+\infty[^{L\times N}, \quad \sum_{\ell=1}^{L} v_j^{(\ell,w)} = u_j^{(w)}, \\ (\forall j \in \{1,\ldots,J\}) & (\forall w \in \mathcal{U}_j) \quad u_j^{(w)} = 1. \end{cases}$$