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Application des grandes matrices aléatoires aux séries temporelles multivariables

Application of large random matrices to multivariate time series analysis

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Resumé

Des techniques issues du domaine des grandes matrices aléatoires ont été récemment utilisées afin d'aborder des problèmes de traitement du signal en grande dimension. Dans leur très grande majorité, les travaux correspondants ont étudié des schémas d'estimation et de détection basés sur des fonctionnelles de la matrice de covariance empirique des observations. La théorie des grandes matrices aléatoires a permis de déterminer le comportement de ces fonctionnelles, et d'en déduire des approches statistiques nouvelles bien adaptées au contexte des grandes dimensions. Cependant, de nombreux problèmes mettant en jeu des séries temporelles de grande dimension font naturellement apparaître des matrices plus générales que les matrices de covariance empirique. Le but de cette thèse est d'étudier les valeurs singulières de deux types de grandes matrices aléatoires jouant un rôle fondamental en statistiques des séries temporelles multivariables, et de déduire des résultats un nouvelle approche permettant d'estimer la dimension minimale des représentations d'état d'un certain type de série temporelle de grande dimension à spectre rationnel. Plus précisément, l'observation est supposée être une version bruitée d'une série temporelle $(u_n)_{n\in\mathbb{Z}}$ de dimension M dont la densité spectrale est rationnelle et de rang déficient, le bruit additif $(v_n)_{n\in\mathbb{Z}}$ étant supposé être blanc, et gaussien complexe de matrice de covariance inconnue. Dans ce contexte, il est tout à fait fondamental d'être capable d'estimer de façon consistante la dimension minimale P des représentations d'état de u à partir des N observations $y_1, y_2, \ldots, y_N$. Si L est n'importe quel entier supposé plus grand que P, les approches les plus traditionnelles sont basées sur le fait que P coïncide avec le rang de la matrice d'autocovariance $R^L_{f|p}$ entre les vecteurs de dimension ML $(y^T_{n+L}, \ldots, y^T_{n+2L-1})^T$ et $(y^T_n, \ldots, y^T_{n+L-1})^T$, mais aussi avec le nombre de valeurs singulières non nulles de la matrice normalisée $C^L = (R^L)^{-1/2}\, R^L_{f|p}(R^L)^{-1/2}$, où R^L représente la matrice de covariance des 2 vecteurs de dimensions ML qui viennent d'être introduits. Dans le régime asymptotique usuel dans lequel $N \to +\infty$ et M et L restent fixes, les matrices $R^L_{f|p}$ et C^L peuvent être estimées de façon consistante par leurs versions empiriques $\hat{R}^L_{f|p}$ et $\hat{C}^L$, et P peut sans difficulté être évalué à partir des plus grandes valeurs singulières de ces estimateurs. Dans le régime des grandes dimensions dans lequel M et N convergent vers $+\infty$ de telle sorte que $c_N = \frac{ML}{N}$ converge vers $0 < c_* \leq 1$, L étant fixe, $\hat{R}^L_{f|p}$ et $\hat{C}^L$ ne sont plus des estimateurs consistants de $R^L_{f|p}$ et C^L au sens de la norme spectrale. Dans ces conditions, il n'est nullement évident qu'il soit toujours possible d'estimer P de façon consistante à partir des valeurs singulières de $\hat{R}^L_{f|p}$ et $\hat{C}^L$. Dans cette thèse, le comportement des valeurs singulières de $\hat{R}^L_{f|p}$ et $\hat{C}^L$ est étudiée dans le régime des grandes dimensions introduit plus haut. Le cas où $u = 0$, ou de façon équivalente $y = v$, est tout d'abord considéré. Il est alors établi que les distributions empiriques des valeurs singulières de $\hat{R}^L_{f|p}$ et $\hat{C}^L$ convergent vers une limite dont les supports $\mathcal{S}_R$ et $\mathcal{S}_C$ sont caractérisés. Il est montré que $\mathcal{S}_C = [0, 2\sqrt{c_*(1-c_*)}] \cup \{1\}\mathbf{1}_{c_*>\frac{1}{2}}$, et que $\mathcal{S}_R$ a une structure plus compliquée. De plus, toutes les valeurs singulières de $\hat{R}^L_{f|p}$ et $\hat{C}^L$ sont situées au voisinage de $\mathcal{S}_R$ et $\mathcal{S}_C$ respectivement. Si u est non nul, la dégénérescence du rang de la densité spectrale de u est utilisée pour étudier si certaines valeurs singulières de $\hat{R}^L_{f|p}$ et $\hat{C}^L$ s'échappent de $\mathcal{S}_R$ et $\mathcal{S}_C$. Il est montré que le nombre de valeurs singulières de $\hat{R}^L_{f|p}$ situées en dehors de $\mathcal{S}_R$ n'est pas directement relié à P, mais que, heureusement, P coïncide avec le nombre de valeurs singulières de $\hat{C}^L$ qui sont plus grandes que $2\sqrt{c_*(1-c_*)}$ si $c_* < \frac{1}{2}$, si le signal u est suffisamment puissant par rapport au bruit v, et si les valeurs singulières non nulles de C^L sont suffisamment grandes. Ces résultats impliquent que les valeurs singulières de $\hat{R}^L_{f|p}$ ne peuvent pas être utilisées pour estimer P de façon consistante dans le régime des grandes dimensions. Par contre, moyennant quelques hypothèses, P peut-être estimé de façon consistante par le nombre de valeurs singulières de $\hat{C}^L$ qui sont plus grandes que $2\sqrt{c_*(1-c_*)}$.

Abstract

A number of recent works proposed to use large random matrix theory in the context of high-dimensional statistical signal processing, traditionally modelled by a double asymptotic regime in which the dimension of the time series and the sample size both grow towards infinity. These contributions essentially addressed detection or estimation schemes depending on functionals of the sample covariance matrix of the observation. Large random matrix theory results were used to evaluate the behaviour of such functionals in the high-dimensional context, and to propose new improved performance approaches. However, fundamental high-dimensional time series problems depend on matrices that are more complicated than the sample covariance matrix. The purpose of the present PhD is to study the behaviour of the singular values of 2 kinds of structured large random matrices, and to use the corresponding results to address an important statistical problem. More specifically, the observation $(y_n)_{n\in\mathbb{Z}}$ is supposed to be a noisy version of a M–dimensional time series $(u_n)_{n\in\mathbb{Z}}$ with rational spectrum that has some particular low rank structure, the additive noise $(v_n)_{n\in\mathbb{Z}}$ being an independent identically distributed sequence of complex Gaussian vectors with unknown covariance matrix. An important statistical problem is the estimation of the minimal dimension P of the state space representations of u from N samples $y_1, \dots, y_N$. If L is any integer larger than P, the traditional approaches are based on the observation that P coincides with the rank of the autocovariance matrix $R^L_{f|p}$ between the ML–dimensional random vectors $(y^T_{n+L}, \dots, y^T_{n+2L-1})^T$ and $(y^T_n, \dots, y^T_{n+L-1})^T$, as well as with the number of non zero singular values of the normalized matrix $C^L = (R^L)^{-1/2}\, R^L_{f|p} (R^L)^{-1/2}$ where R^L represents the covariance matrix of the above ML–dimensional vectors. In the low-dimensional regime where $N \to +\infty$ while M and L are fixed, the matrices $R^L_{f|p}$ and C^L can be consistently estimated by their empirical counterparts $\hat{R}^L_{f|p}$ and $\hat{C}^L$, and P can be evaluated from the largest singular values of $\hat{R}^L_{f|p}$ and $\hat{C}^L$. If however M and N converge towards $+\infty$ in such a way that $c_N = \frac{ML}{N}$ converges towards $0 < c_* \leq 1$, L being fixed, the above estimates $\hat{R}^L_{f|p}$ and $\hat{C}^L$ do not converge towards their true values in the spectral norm sense. It is therefore not obvious whether the largest singular values of $\hat{R}^L_{f|p}$ and $\hat{C}^L$ can be used in order to estimate P consistently. In this thesis, the behaviour of the singular values of $\hat{R}^L_{f|p}$ and $\hat{C}^L$ in the above high-dimensional regime are studied. The case where $u = 0$, or equivalently $y = v$, is first considered and it is established that the empirical singular values distribution of $\hat{R}^L_{f|p}$ and $\hat{C}^L$ converge towards a limit. The supports $\mathcal{S}_R$ and $\mathcal{S}_C$ of the corresponding limit distributions are characterized : it is proved that $\mathcal{S}_C = [0, 2\sqrt{c_*(1-c_*)}] \cup \{1\}\mathbf{1}_{c_* > \frac{1}{2}}$ and that the structure of $\mathcal{S}_R$ is more intricate. It is moreover established that all the singular values of $\hat{R}^L_{f|p}$ and $\hat{C}^L$ are located in the neighbourhood of $\mathcal{S}_R$ and $\mathcal{S}_C$ respectively. When u is present, the low rank structure of u is used in order to study whether some singular values of $\hat{R}^L_{f|p}$ and $\hat{C}^L$ escape from $\mathcal{S}_R$ and $\mathcal{S}_C$. It is shown that the number of singular values of $\hat{R}^L_{f|p}$ located outside $\mathcal{S}_R$ is not directly related to P, while, fortunately, P coincides with the number of singular values of $\hat{C}^L$ that are larger than $2\sqrt{c_*(1-c_*)}$, provided $c_* < \frac{1}{2}$, the signal u is powerful enough compared to the noise and the non zero singular values of C^L are large enough. These results imply that while the singular values of $\hat{R}^L_{f|p}$ can be used in order to estimate P consistently in the standard low-dimensional regime, this is no longer the case in the high-dimensional context considered here. Fortunately, under certain assumptions, P can still be consistently estimated as the number of singular values of $\hat{C}^L$ that are larger than $2\sqrt{c_*(1-c_*)}$

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Chapitre 1

Introduction.

Due to the spectacular development of data acquisition devices and sensor networks, it becomes very common to be faced with high-dimensional time series in various fields such as digital communications, environmental sensing, electroencephalography, analysis of financial datas, industrial monitoring, ... In this context, it is not always possible to collect a large enough number of observations to perform statistical inference because the durations of the signals are limited and/or because their statistics are not time-invariant over large enough temporal windows. As a result, fundamental inference schemes do not behave as in the classical low-dimensional regimes. This stimulated considerably in the ten past years the development of new statistical approaches aiming at mitigating the above mentioned difficulties. In particular, a number of works proposed to use large random matrix theory in the context of high-dimensional statistical signal processing, traditionally modelled by a double asymptotic regime in which the dimension of the time series and the sample size both grow towards infinity. These contributions essentially addressed detection or estimation schemes depending on functionals of the sample covariance matrix of the observation. Large random matrix theory results were used to evaluate the behaviour of such functionals in the high-dimensional context, and to propose new improved performance approaches. However, fundamental high-dimensional time series problems depend on matrices that are more complicated than the sample covariance matrix. The purpose of the present PhD is to study the behaviour of the eigenvectors of 2 kinds of structured large random matrices, and to use the corresponding results to address certain important statistical signal processing problems.

1.1 Motivation

In this work we consider a M–dimensional multivariate time series $(y_n)_{n\in\mathbb{Z}}$ generated as

$$y_n = u_n + v_n, \tag{1.1}$$

where $(v_n)_{n\in\mathbb{Z}}$ is a complex Gaussian "noise" term such that $\mathbb{E}(v_{n+k}v_n^*) = R\delta_k$ for some unknown positive definite matrix R, and where $(u_n)_{n\in\mathbb{Z}}$ is a "useful" non observable Gaussian signal with rational spectrum. Thus, u_n can be represented as

$$x_{n+1} = Ax_n + B\omega_n,\ u_n = Cx_n + D\omega_n, \tag{1.2}$$

where $(\omega_n)_{n\in\mathbb{Z}}$ is a $K \leq M$–dimensional white noise sequence ($\mathbb{E}(\omega_{n+k}\omega_n^*) = I_K\,\delta_k$), A is a deterministic $P\times P$ matrix whose spectral radius $\rho(A)$ is strictly less than 1, and where B, C, D are deterministic matrices. The P-dimensional Markovian sequence $(x_n)_{n\in\mathbb{Z}}$ is called the state-space sequence associated to (1.2). The state space representation (1.2) is said to be minimal if the dimension P of the state space sequence is minimal. Given the autocovariance sequence $(R_{u,n})_{n\in\mathbb{Z}}$ of u (i.e. $R_{u,n} = \mathbb{E}(u_{k+n}u_k^*)$ for each n), the so-called stochastic realization problem of $(u_n)_{n\in\mathbb{Z}}$ consists in characterizing all the minimal state space representations (1.2) of u, or equivalently in identifying all the minimum Mac-Millan degrees [1] matrix-valued functions $\Phi(z) = D + C(zI - A)^{-1}B$ such that $\rho(A) < 1$ and

$$S_u(e^{2i\pi f}) = \sum_{n\in\mathbb{Z}} R_{u,n}e^{-2i\pi nf} = \Phi(e^{2i\pi f})\Phi(e^{2i\pi f})^* \tag{1.3}$$

1. The Mac-Millan degree of a rational matrix-valued function Φ is defined as the minimal dimension of the matrices A for which $\Phi(z)$ can be represented as $D + C(zI - A)^{-1}B$

for each f. Such a function Φ is called a minimal degree causal spectral factorization of S_u. We refer the reader to [30] or [45] for more details.

The identification of P and of matrices C and A is based on the observation that the autocovariance sequence of u can be represented as

$$R_{u,n} = \mathbb{E}(u_{n+k}u_k^*) = CA^{n-1}G \tag{1.4}$$

for each $n \geq 1$, where the 3 matrices (A, C, G) are unique up to similarity transforms, thus showing that the matrices C and A associated to a minimal realization are uniquely defined (up to a similarity). Moreover, if we define the autocovariance matrix $R^L_{f|p,u}$ between the past and the future of u as

$$R^L_{f|p,u} = \mathbb{E}\left[\begin{pmatrix} u_{n+L} \\ u_{n+L+1} \\ \vdots \\ u_{n+2L-1} \end{pmatrix} \left(u_n^*, u_{n+1}^*, \ldots, u_{n+L-1}^*\right)\right] \tag{1.5}$$

then, it holds that

$$R^{(L)}_{f|p,u} = \mathcal{O}^{(L)}\,\mathcal{C}^{(L)}, \tag{1.6}$$

where matrix $\mathcal{O}^{(L)}$ is the $ML \times P$ "observability" matrix

$$\mathcal{O}^{(L)} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{pmatrix} \tag{1.7}$$

and matrix $\mathcal{C}^{(L)}$ is the $P \times ML$ "controllability" matrix

$$\mathcal{C}^{(L)} = \left(A^{L-1}G, A^{L-2}G, \ldots, G\right). \tag{1.8}$$

For each $L \geq P$, the rank of $R^{(L)}_{f|p,u}$ remains equal to P, and each minimal rank factorization of $R^{(L)}_{f|p,u}$ can be written as (1.6) for some particular triple (A, C, G). In particular, if $R^{(L)}_{f|p,u} = \Theta\Gamma\tilde{\Theta}^*$ is the singular value decomposition of $R^{(L)}_{f|p,u}$, matrix $\Theta\Gamma^{1/2}$ coincides with the observability matrix $\mathcal{O}^{(L)}$ of a pair (C, A). C and A are immediately obtained from the knowledge of the structured matrix $\mathcal{O}^{(L)}$. This discussion shows that the evaluation of P, C and A from the autocovariance sequence of u is an easy problem. We mention that, while C and A are essentially unique, there exist in general more than one pair (B, D) for which (1.2) holds because the minimal degree spectral factorization problem (1.3) has more than 1 solution. We refer the reader to [30] or [45].

We notice that as $(v_n)_{n\in\mathbb{Z}}$ in (1.1) is an uncorrelated sequence, it holds that $R_{y,n} = \mathbb{E}(y_{n+k}y_k^*)$ coincides with $R_{u,n}$ for each $n \geq 1$. Therefore, P also coincides with the minimal dimension of state-space realizations of y, and matrices C and A can still be identified from the autocovariance sequence of the noisy version y of u. In practice, however, the exact autocovariance sequence $(R_{y,n})_{n\geq 1}$ is in general unknown, and it is necessary to estimate P and (C, A) from the sole knowledge of N samples $y_1 = u_1 + v_1, y_2 = u_2 + v_2, \ldots, y_N = u_N + v_N$. For this, P is first estimated as the number of significant singular values of the empirical estimate $\hat{R}^L_{f|p,y}$ of the true matrix $R^L_{f|p,y} = R^L_{f|p,u}$ defined by

$$\hat{R}^L_{f|p,y} = \frac{Y_{f,N}Y_{p,N}^*}{N},$$

where matrices $Y_{f,N}$ and $Y_{p,N}$ defined as

$$Y_{p,N} = \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-1} & y_N \\ y_2 & y_3 & \cdots & y_N & y_{N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_L & y_{L+1} & \cdots & y_{N+L-2} & y_{N+L-1} \end{pmatrix} \tag{1.9}$$

and

$$Y_{f,N} = \begin{pmatrix} y_{L+1} & y_{L+2} & \cdots & y_{N-1+L} & y_{N+L} \\ y_{L+2} & y_{L+3} & \cdots & y_{N+L} & y_{N+L+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{2L} & y_{2L+1} & \cdots & y_{N+2L-2} & y_{N+2L-1} \end{pmatrix}. \tag{1.10}$$

We note that the samples $(y_{N+l})_{l=1,\ldots,2L-1}$ are supposed to be available while we have assumed that only the first N samples are observed. In order to simplify the presentation, this end effect is neglected. If $(\hat{\gamma}_p)_{p=1,\ldots,P}$ and $\hat{\Theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_P)$ are the P largest singular values and corresponding left singular vectors of matrix $\hat{R}^{(L)}_{f|p,y}$, and if $\hat{\Gamma}$ is the $P \times P$ diagonal matrix with diagonal entries $(\hat{\gamma}_p)_{p=1,\ldots,P}$, the $ML \times P$ matrix $\hat{\mathcal{O}}^{(L)} = \hat{\Theta}\hat{\Gamma}^{1/2}$ is an estimator of an observability matrix $\mathcal{O}^{(L)}$. $\hat{\mathcal{O}}^{(L)}$ has not necessarily the structure of an observability matrix, but C and A can be estimated respectively by the top $M \times P$ block $\hat{C}$ of $\hat{\mathcal{O}}^{(L)}$ and by the argument $\hat{A}$ of the minimum of the quadratic fuction

$$\mathrm{Tr}\left(\left(\hat{\mathcal{O}}^{(L)}_{\text{down}} A - \hat{\mathcal{O}}^{(L)}_{\text{up}} \right) \left(\hat{\mathcal{O}}^{(L)}_{\text{down}} A - \hat{\mathcal{O}}^{(L)}_{\text{up}} \right)^* \right),$$

where the operator "down" (resp. "up") suppresses the last (resp. the first) M rows from $ML \times P$ matrix $\hat{\mathcal{O}}^{(L)}$. This approach provides a consistent estimate of P, C, A when $N \to +\infty$ while M, L and P are fixed parameters. We refer the reader to [12] for a detailed analysis of this statistical inference scheme known as the Principal Component Algorithm.

If M is large and that the sample size N cannot be arbitrarily larger than M, the ratio ML/N may not be small enough to make reliable the above statistical analysis. It is thus relevant to study the behaviour of the above estimators in asymptotic regimes where M and N both converge towards $+\infty$ in such a way that $\frac{ML}{N}$ converges towards a non zero constant. In this context, matrix $\hat{R}^{(L)}_{f|p,y}$ is no longer a consistent estimate of the true matrix $R^{(L)}_{f|p,y}$ in the spectral norm sense. Therefore, the singular values of $\hat{R}^{(L)}_{f|p,y}$ have no reasons to behave as those of $R^{(L)}_{f|p,y}$. Thus, it appears of fundamental interest to study the behaviour of the singular values of $\hat{R}^{(L)}_{f|p,y}$, and to study whether its largest singular still allow to estimate P consistently, at least if the useful signal u appears as powerful enough compared to the noise v. The behaviour of the associated singular vectors would of course be of potential interest in order to address the estimation of matrices C and A, but this important topic is not addressed in this thesis.

Another way to estimate P is to resort to the concept of canonical correlation coefficients between the past and the future of the time series $(y_n)_{n\in\mathbb{Z}}$. We denote by $\mathcal{Y}_p$ and $\mathcal{Y}_f$ the (infinite dimensional) subspaces generated by the components of $(y_n)_{n\leq 0}$ and $(y_n)_{n\geq 1}$, and consider 2 orthonormal bases $(\omega_{p,k})_{k\geq 0}$ and $(\omega_{f,k})_{k\geq 0}$ of $\mathcal{Y}_p$ and $\mathcal{Y}_f$ respectively. Then the canonical correlation coefficients between the past and future of y are defined as the singular values of the (infinite) matrix with entries $\mathbb{E}(\omega_{f,k}\omega^*_{p,l})$ (see [25] for more informations), and it is well known that P coincides with the number on non zero such coefficients. See [30] for an exhaustive presentation of the related results and their important implications on questions such as the identification of the state space models or on reduction model technics. Moreover, if $\mathcal{Y}_p$ and $\mathcal{Y}_f$ are replaced by the finite dimensional spaces $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ generated respectively by the components of $y_n, n = -(L-1), \ldots, 0$ and $y_n, n = 1, \ldots, L$ for a certain integer $L \geq P$, it turns out that the number of non zero canonical correlation coefficients between $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ is still equal to P. We refer again to [30] for more details on the effects of the truncation. In order to estimate P from the N avalaible observations $y_1, \ldots, y_N$ in the standard low-dimensional regime $N \to +\infty$ while M and L are fixed, a standard solution is to estimate the correlation coefficients between $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ by the canonical correlation coefficients between the row spaces of matrices $Y_{p,N}$ and $Y_{f,N}$ defined by (1.9) and (1.10) respectively, and to estimate P as the number of significant coefficients, i.e. as the number of significant singular values of matrix $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1/2}$, or equivalently of the number of significant eigenvalues of $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1}\hat{R}^{L*}_{f|p,y}(\hat{R}^L_{f,y})^{-1/2}$. Here, matrices $\hat{R}^L_{f,y}$ and $\hat{R}^L_{p,y}$ are defined by $\hat{R}^L_{f,y} = \frac{Y_{f,N}Y^*_{f,N}}{N}$ and $\hat{R}^L_{p,y} = \frac{Y_{p,N}Y^*_{p,N}}{N}$ respectively. In the low-dimensional

regime, this approach provides consistent estimates of P, but this is no longer the case in the high-dimensional regime where M and N both converge towards $+\infty$ in such a way that $\frac{ML}{N}$ converges towards a non zero constant. The study of the eigenvalues of $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1}\hat{R}^{L*}_{f|p,y}(\hat{R}^L_{f,y})^{-1/2}$ in the above high-dimensional regime thus appears as a highly relevant problem.
Without formulating specific assumptions on u, these problems seem very complicated. In the past, a number of works addressed high-dimensional inference schemes based on the eigenvalues and eigenvectors of the empirical covariance matrix of the observation (see e.g. [37], [35], [38], [20], [47], [48], [13], [44]) when the useful signal lives in a low-dimensional deterministic subspace. Using results related to spiked large random matrix models (see e.g. [4] [5], [41]), based on perturbation technics, a number of important statistical problems could be addressed using large random matrix theory technics. In this thesis, we follow the same kind of approach to address the estimation problem of P when u satisfies some low rank assumptions.

1.2 Contribution of the thesis.

Time series y being the sum of the noise v with a useful signal u is generated by certain state-space models (1.2) to be precised below, the general topic of the thesis is to study the singular values of the empirical estimates $\hat{R}^L_{f|p,y}$ and $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1/2}$ in the following asymptotic regime :

M and N both converge towards $+\infty$ in such a way that $c_N = \frac{ML}{N} \to c_*$, L being a fixed integer.

When this will be possible, we will deduce from the corresponding results conditions under which P can be consistently estimated from the above matrices. We notice that, as L is a fixed integer, M and N are thus of the same order to magnitude. However, it should be mentioned that the case where both M and L converge towards $+\infty$ in such a way that $c_N \to c_*$ is also of potential interest. While a number of results of this thesis obtained in the absence of signal (i.e. $y = v$) could be generalized to this context, the study of the largest singular values of $\hat{R}^L_{f|p,y}$ and $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1/2}$ in the presence of signal would be deeply modified because, in contrast with the case L finite, matrices $Y_{p,N}$ and $Y_{f,N}$ would not be finite rank perturbations of the matrices $V_{p,N}$ and $V_{f,N}$ defined from the noise samples $v_1, \ldots, v_{N+2L-1}$. This thesis is thus only devoted to the above high-dimensional regime with L finite.

This thesis is structured as follows.

In **Chapter 2**, we present some basic tools and notations that are used along the thesis.

Chapter 3 is dedicated to the study of the singular values of matrix $\hat{R}^L_{f|p,y}$, or equivalently of the eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$, in the case when the signal is absent, i.e. $y_n = v_n$. In this context, it thus holds that $\frac{1}{\sqrt{N}}Y_{p,N} = \frac{1}{\sqrt{N}}V_{p,N}$ and $\frac{1}{\sqrt{N}}Y_{f,N} = \frac{1}{\sqrt{N}}V_{f,N}$. In the following, we denote by $W_{p,N}$ and $W_{f,N}$ the normalized matrices

$$W_{p,N} = \frac{1}{\sqrt{N}}V_{p,N},\ W_{f,N} = \frac{1}{\sqrt{N}}V_{f,N} \tag{1.11}$$

The goal of this chapter is to study the almost sure location of the eigenvalues of $W_{f,N}W^*_{p,N}\ W_{p,N}W^*_{f,N}$ in the above asymptotic regime.

For this, we first evaluate the behaviour of the empirical eigenvalue distribution $\hat{\nu}_N$ of the $ML \times ML$ matrix $W_{f,N}W^*_{p,N}W_{p,N}W^*_{f,N}$. Using Gaussian tools, i.e. integration by parts formula in conjunction with the Poincaré-Nash inequality (see e.g. [40]), we characterize the asymptotic behaviour of the resolvent $Q_N(z) = (W_{f,N}W^*_{p,N}W_{p,N}W^*_{f,N} - zI)^{-1}$. As the entries of $W_{f,N}W^*_{p,N}W_{p,N}W^*_{f,N}$ are biquadratic functions of $y_1, \ldots, y_{N+2L-1}$, we rather use the well-known linearization trick that consists in studying the resolvent $\mathbf{Q}_N(z)$ of the $2ML \times 2ML$ hermitized version

$$\begin{pmatrix} 0 & W_{f,N}W^*_{p,N} \\ W_{p,N}W^*_{f,N} & 0 \end{pmatrix}$$

of matrix $W_{f,N}W^*_{p,N}$. As is well known, the first $ML \times ML$ diagonal block of $\mathbf{Q}_N(z)$ coincides with $zQ_N(z^2)$. Therefore, we characterize the asymptotic behaviour of $\mathbf{Q}_N(z)$, and deduce from this the results concerning

$Q_N(z)$. The hermitized version is this time a quadratic function of $y_1, \dots, y_{N+2L-1}$, and the Gaussian calculus that is needed in order to study $\mathbf{Q}_N(z)$ appears much simpler than if $Q_N(z)$ was evaluated directly.

We introduce the $M \times M$ matrix-valued function $T_N(z)$ defined by

$$T_N(z) = -\left(zI_M + \frac{zc_N t_N(z)}{1 - zc_N^2 t_N^2(z)} R_N\right)^{-1},$$

$t_N(z)$ being the unique solution of the equation

$$t_N(z) = \frac{1}{M} \mathrm{Tr} R_N \left(-zI_M - \frac{zc_N t_N(z)}{1 - zc_N^2 t_N^2(z)} R_N\right)^{-1}$$

such that $t_N(z)$ and $zt_N(z)$ belong to $\mathbb{C}^+$ when $z \in \mathbb{C}^+$. $t_N(z)$ and $T_N(z)$ are shown to coincide with the Stieltjes transforms of a scalar measure μ_N and of a $M \times M$ positive matrix valued measure ν_N^T respectively (see Section 3.3 for a formal definition of a $M \times M$ positive matrix valued measure). We recall that $R_N = \mathbb{E}(v_n v_n^*)$ is the covariance matrix of the random vectors $(v_n)_{n \in \mathbb{Z}}$. It is shown that the resolvent $Q_N(z)$ of $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ has in some sense the same asymptotic behaviour than $I_L \otimes T_N(z)$. Moreover, recalling that $\hat{\nu}_N$ denotes the empirical eigenvalue distribution of $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$, it is proved that $\nu_N = \frac{1}{M} \mathrm{Tr}(\nu_N^T)$ is a probability measure such that $\hat{\nu}_N - \nu_N \to 0$ weakly almost surely. ν_N is referred to as the deterministic equivalent of $\hat{\nu}_N$. We study the properties and the support of ν_N, or equivalently of μ_N because the 2 measures are absolutely continuous one with respect to each other. For this, we study the behaviour of $t_N(z)$ when z converges towards the real axis. For each $x > 0$, the limit of $t_N(z)$ when $z \in \mathbb{C}^+$ converges towards x exists and is finite. If $c_N \leq 1$, we deduce from this that ν_N is absolutely continuous w.r.t. the Lebesgue measure. The corresponding density $g_N(x)$ is real analytic on $\mathbb{R}^{+*}$, and converges towards $+\infty$ when $x \to 0, x > 0$. If $c_N < 1$, it holds that $g_N(x) = \mathcal{O}(\frac{1}{\sqrt{x}})$ while $g_N(x) = \mathcal{O}(\frac{1}{x^{2/3}})$ if $c_N = 1$. If $c_N > 1$, ν_N contains a Dirac mass at 0 with weight $1 - \frac{1}{c_N}$ and an absolutely continuous component. In order to analyse the support of μ_N and ν_N, we establish that the function $w_N(z)$ defined by

$$w_N(z) = zc_N t_N(z) - \frac{1}{c_N t_N(z)}$$

is solution of the equation $\phi_N(w_N(z)) = z$ for each $z \in \mathbb{C} - \mathbb{R}^+$ where $\phi_N(w)$ is the function defined by

$$\phi_N(w) = c_N w^2 \frac{1}{M} \mathrm{Tr} R_N \left(R_N - wI\right)^{-1} \left(c_N \frac{1}{M} \mathrm{Tr} R_N \left(R_N - wI\right)^{-1} - 1\right).$$

This property allows to prove that apart $\{0\}$ when $c_N > 1$, the support of μ_N is a union of intervals whose end points are the extrema of ϕ_N whose arguments verify $\frac{1}{M} \mathrm{Tr} R_N \left(R_N - wI\right)^{-1} < 0$. A sufficient condition on the eigenvalues of R_N ensuring that the support of μ_N is reduced to a single interval is formulated. Using the Haagerup-Thorbjornsen approach ([17]), it is moreover proved in section 3.7 that for each N large enough, all the eigenvalues of $W_{f,N}\ W_{p,N}^*\ W_{p,N}\ W_{f,N}^*$ lie in a neighbourhood of the support of the deterministic equivalent ν_N.

We finally indicate that the use of free probability tools is an alternative approach to characterize the asymptotic behaviour of $\hat{\nu}_N$. While this approach is simpler than the use of the Gaussian tools proposed in the present chapter, we mention that the above free probability theory arguments do not allow to study the asymptotic behaviour of the resolvent of $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ needed to address in Chapter 4 the case where a useful signal u is present.

In **Chapter 4** we pass to the case when the signal is present and study its influence on the eigenvalues of matrix $\hat{R}_{f|p,y}^L (\hat{R}_{f|p,y}^L)^*$. For this, we use a classical approach based on the observation that matrix $\frac{Y_f Y_p^*}{N}$ is a finite rank perturbation of matrix $\frac{V_f V_p^*}{N}$ due to the noise $(v_n)_{n \in \mathbb{Z}}$. For simplicity we assume that the support of μ_N is reduced to one interval denoted $[0, x_{N,+}]$. We first present assumptions on the useful signal as well as on the asymptotic behaviour of the empirical eigenvalue distribution of the covariance matrix R_N

of the noise. These assumptions ensure that a number of terms depending on N have finite limits when $N \to +\infty$, and allow to prove that some of the largest eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$ may converge towards limits that are located outside the "bulk" $[0, x_{N,+}]$ for each N large enough. More precisely, we assume that the dimension P of the minimal state space representation (1.2) of u is a fixed integer P, that K is also fixed, and that the matrices A and B do not scale as well with M and N. Therefore, the P–dimensional markovian sequence x does not depend on M and N. If we denote by u^L_n the ML–dimensional vector $u^L_n = (u^T_n, \dots, u^T_{n+L-1})^T$, it is easily seen that the covariance matrix $R^L_{u,N}$ on u^L_n is rank deficient, and that its rank r verifies $P \leq r \leq P + KL$. While r may depend on M, N, we assume that it is constant for N large enough. If we introduce the eigenvalue / eigenvector decomposition of $R^L_{u,N} = \Theta_N \Delta^2_N \Theta^*_N$, it is easily seen that $\frac{U_{i,N}}{\sqrt{N}}$ is a rank r matrix for each N large enough, and if

$$\frac{U_{i,N}}{\sqrt{N}} = \Theta_{i,N} \Delta_{i,N} \tilde{\Theta}^*_{i,N}$$

represents its singular value decomposition, then we infer that $\|\Theta_{i,N}\Theta^*_{i,N} - \Theta_N \Theta^*_N\| \to 0$ and $\|\Delta_{i,N} - \Delta_N\| \to 0$. Moreover, for N large enough, matrix $\tilde{\Theta}^*_{f,N}\tilde{\Theta}_{p,N}$ is a rank P matrix whose singular values are the canonical correlation coefficients between the row spaces of $U_{p,N}$ and $U_{f,N}$, and when $N \to +\infty$, these coefficients have the same asymptotic behaviour than the canonical correlation coefficients between the spaces generated by the components of u^L_n and of u^L_{n+L}. In the following, we assume that the diagonal matrices Δ_N converges towards a limit matrix $\Delta_* > 0$ when $N \to +\infty$, or equivalently that the r non zero eigenvalues of $R^L_{u,N}$ converge towards non zero limits, and that the $r \times r$ rank P matrices $\tilde{\Theta}^*_{f,N}\tilde{\Theta}_{p,N}$ converge towards a (necessarily rank P) matrix Ω_*. The empirical eigenvalue distribution of R_N is also assumed to have a compactly supported limit distribution carried by the interval $[\lambda_{-,*}, \lambda_{+,*}]$ where $\lambda_{-,*}$ and $\lambda_{+,*}$ are the limits when $N \to +\infty$ of the smallest and of the largest eigenvalues of R_N assumed to exist. This imply that the sequence $(\mu_N)_{N \geq 1}$ converges towards a limit μ_*, and that the Stieltjes transforms sequence $(t_N)_{N\geq 1}$ converges towards a limit $t_*(z)$. Moreover, $x_{N,+}$ converges towards a finite limit $x_{+,*}$. Finally, under a certain extra assumption that will be stated inside the Chapter, the $r \times r$ matrix valued measure β_N defined by

$$d\beta_N(\lambda) = \Theta^*_N \left(I_L \otimes d\nu^T_N(\lambda)\right) \Theta_N \tag{1.12}$$

is shown to converge towards a limit β_*. We are now in position to define the matrix-valued function $H_*(z)$ defined by

$$H_*(z) = \begin{pmatrix} \frac{c_* \mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta^2_* - (\mathbf{T}_{\beta_*}(z))^{-1} & \frac{\Delta_*\Omega^*_*\Delta_*}{(1-(c_*\mathbf{t}_*(z))^2} \\ \frac{\Delta_*\Omega_*\Delta_*}{(1-(c_*\mathbf{t}_*(z))^2} & \frac{c_* \mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta^2_* - (\mathbf{T}_{\beta_*}(z))^{-1} \end{pmatrix}$$

where $\mathbf{t}_*(z)$ is defined by $\mathbf{t}_*(z) = zt_*(z^2)$ and $\mathbf{T}_{\beta_*}(z) = zT_{\beta_*}(z^2)$, $T_{\beta_*}(z)$ being the Stieltjes transform of β_*. Then, we establish that the equation

$$\det\left(H_*(y)\right) = 0$$

has s solutions that are larger than $\sqrt{x_{*,+}}$, where s is an integer verifying $0 \leq s \leq 2r$. Moreover, for N large enough, exactly s eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$ escape from the interval $[0, x_{*,+}]$, and converge towards the squares of the s solutions of $\det\left(H_*(y)\right) = 0$ that are larger than $\sqrt{x_{*,+}}$. While it is difficult to characterise s in the general case, we exhibit examples where P takes the same value, but s can takes any value of $\{0, 1, \dots, 2r\}$. This means that the number of eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$ that escape from $[0, x_{*,+}]$ does not in general coincide with P, and it seems not obvious to estimate P consistently from the largest eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$.

In **Chapter 5**, under the same kind of assumptions on the useful signal, we establish that in contrast with $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$, it is possible to estimate P consistently from the largest eigenvalues of the matrix $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1}\hat{R}^{L*}_{f|p,y}(\hat{R}^L_{f,y})^{-1/2}$ provided the useful signal is powerful enough and its own correlation coefficients between the past and the future are large enough. In the following, in order to simplify the notations, we introduce matrix

$$\Sigma_{i,N} = \frac{Y_{i,N}}{\sqrt{N}} = \frac{V_{i,N}}{\sqrt{N}} + \frac{U_{i,N}}{\sqrt{N}} = W_{i,N} + \frac{U_{i,N}}{\sqrt{N}}$$

for $i = \{p, f\}$ (we recall that $W_{p,N}$ and $W_{f,N}$ are defined by (1.11)), and remark that

$$(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1}\hat{R}^{L*}_{f|p,y}(\hat{R}^L_{f,y})^{-1/2} = (\Sigma_{f,N}\Sigma^*_{f,N})^{-1/2}\Sigma_{f,N}\Sigma^*_{p,N}(\Sigma_{p,N}\Sigma^*_{p,N})^{-1}\Sigma_{p,N}\Sigma^*_{f,N}(\Sigma_{f,N}\Sigma^*_{f,N})^{-1/2}$$

It is clear that apart 0, the eigenvalues of this matrix coincide with the eigenvalues of the matrix

$$\Sigma^*_{p,N}(\Sigma_{p,N}\Sigma^*_{p,N})^{-1}\Sigma_{p,N}\Sigma^*_{f,N}(\Sigma_{f,N}\Sigma^*_{f,N})^{-1}\Sigma_{f,N} = \Pi_{p,N}\Pi_{f,N}$$

where $\Pi_{p,N} = \Sigma^*_{p,N}(\Sigma_{p,N}\Sigma^*_{p,N})^{-1}\Sigma_{p,N}$ and $\Pi_{f,N} = \Sigma^*_{f,N}(\Sigma_{f,N}\Sigma^*_{f,N})^{-1}\Sigma_{f,N}$ represent the orthogonal projection matrices over the row spaces of $Y_{p,N}$ and $Y_{f,N}$ respectively. Since the eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ do not exceed 1, it is natural to assume in this part that $c_N = ML/N \to c_* \in (0, 1]$.

In the absence of signal.

We first analyse the eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ when the signal is absent, i.e. $y = v$, or equivalently $\frac{Y_{p,N}}{\sqrt{N}} = \Sigma_{p,N} = W_{p,N}$ and $\frac{Y_{f,N}}{\sqrt{N}} = \Sigma_{f,N} = W_{f,N}$. We notice that for each n, vector v_n can be written as $v_n = R_N^{1/2} v_n^{iid}$ where the vectors $(v_n^{iid})_{n\in\mathbb{Z}}$ are independent and $\mathcal{N}_c(0, I)$ distributed. Then, it holds that $W_{i,N} = R_N^{1/2} W^{iid}_{i,N}$ for $i = p, f$ where matrices $W^{iid}_{p,N}$ and $W^{iid}_{f,N}$ are built from the $\mathcal{N}_c(0, I)$ distributed vectors $(v_n^{iid})_{n=1,\dots,N+2L-1}$. As the row spaces of $W_{i,N}$ and $W^{iid}_{i,N}$ coincide, the two projectors $\Pi_{p,N}$ and $\Pi_{f,N}$ coincide with the projectors $\Pi^{iid}_{p,N}$ and $\Pi^{iid}_{f,N}$ defined from $W^{iid}_{p,N}$ and $W^{iid}_{f,N}$. Therefore, when the useful signal is absent, we can assume without restriction that $R_N = I$.

In Chapter 5, we denote by $\hat{\nu}_N$ the empirical eigenvalue distribution of $\Pi_{p,N}\Pi_{f,N}$ despite the fact that $\hat{\nu}_N$ represents in Chapter 3 the empirical eigenvalue distribution of $W_{f,N}W^*_{p,N}$. If matrices $W_{p,N}$ and $W_{f,N}$ where mutually independent random matrices with i.i.d. complex standard Gaussian entries, free probability theory methods (see e.g. [50]) or Gaussian tools ([46]) would imply that if $\tilde{\nu}_N$ denotes the free multiplicative convolution product of $c_N\delta_1 + (1 - c_N)\delta_0$ with itself, then, $\hat{\nu}_N - \tilde{\nu}_N \to 0$ almost surely. As it is well known, $\tilde{\nu}_N$ is given by

$$d\tilde{\nu}_N(\lambda) = \frac{\sqrt{\lambda(4c_N(1 - c_N) - \lambda)}}{2\pi\lambda(1 - \lambda)}\mathbf{1}_{[0,4c_N(1-c_N)]}d\lambda + (1 - c_N)\delta_\lambda + \max(2c_N - 1, 0)\delta_{\lambda-1} \tag{1.13}$$

and its Stieltjes transform, denoted $\tilde{t}_N(z)$, is equal to

$$\tilde{t}_N(z) = \frac{z - 2(1 - c_N) + \sqrt{z(z - 4c_N(1 - c_N))}}{2(1 - z)z} \tag{1.14}$$

for each $z \in \mathbb{C}^+$, where we define function $z \mapsto \sqrt{z}$ for $z = |z|e^{i\theta}$, $\theta \in (0, 2\pi)$ as $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$. Moreover, for each $\epsilon > 0$, almost surely, for N large enough, all the eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ strictly less than 1 would belong to $[0, 4c_*(1 - c_*) + \epsilon]$. In our context, the structured matrices $W_{p,N}$ and $W_{f,N}$ are not mutually independent, and their elements are not i.i.d. However, we establish that the above results remain true. For this, we use the Stieltjes transform approach and evaluate the asymptotic behaviour of the resolvent of matrix $\Pi_{p,N}\Pi_{f,N}$. In order to be able to use Gaussian tools, the matrices $\Pi_{p,N}$ and $\Pi_{f,N}$ should be differentiable functions of the entries of matrices $W_{p,N}$ and $W_{f,N}$ respectively. However, this is not the case when these entries are such that $W_{p,N}W^*_{p,N}$ or $W_{f,N}W^*_{f,N}$ are not invertible. In order to address this technical problem, we use a regularization scheme introduced in [21]. We introduce η_N defined by

$$\eta_N = \det\,[\phi(W_{f,N}W^*_{f,N})]\det\,[\phi(W_{p,N}W^*_{p,N})],$$

where ϕ is a smooth function such that

$$\phi(\lambda) = \begin{cases} 1, \text{ for } \lambda \in [(1 - \sqrt{c_*})^2 - \epsilon],\ [(1 + \sqrt{c_*})^2 + \epsilon], \\ 0, \text{ for } \lambda \in [-\infty,\ (1 - \sqrt{c_*})^2 - 2\epsilon] \cup [(1 + \sqrt{c_*})^2 + 2\epsilon,\ +\infty] \end{cases}$$

and $\phi(\lambda) \in (0, 1)$ elsewhere. As $\eta_N = 0$ if $W_{i,N}W^*_{i,N}$ is not invertible for $i = p$ or $i = f$, $\eta_N\Pi_{i,N}$, considered as a function of the entries of $W_{p,N}$ and $W_{f,N}$, is a differentiable function whose derivatives are bounded polynomially. Moreover, it is shown in [32] that the empirical eigenvalue distribution of $W_{i,N}W_{i,N}^*$ for $i = \{p,f\}$ converges towards the Marcenko-Pastur distribution, and that almost surely, for N greater than a random integer, its eigenvalues located in the neighbourhood of $[(1-\sqrt{c_*})^2, (1+\sqrt{c_*})^2]$. Therefore, almost surely, for N large enough, $\eta_N = 1$, and $\eta_N \Pi_{i,N} = \Pi_{i,N}$ for $i = \{p,f\}$. The almost sure behaviours of the resolvents $(\Pi_{p,N}\Pi_{f,N} - zI)^{-1}$ and $(\eta_N \Pi_{p,N} \eta_N \Pi_{f,N} - zI)^{-1}$ thus coincide. In this chapter, we thus study the resolvent $Q_N(z) = (\eta_N \Pi_{p,N} \eta_N \Pi_{f,N} - zI)^{-1}$ using the integration by parts formula and the Nash-Poincaré inequality (we again use in Chapter 5 a notation used to denote a different object in Chapter 3). As in Chapter 3, we rather evaluate the behaviour of the resolvent

$$\mathbf{Q}_N(z) = \begin{pmatrix} \mathbf{Q}_{N,pp}(z) & \mathbf{Q}_{N,pf}(z) \\ \mathbf{Q}_{N,fp}(z) & \mathbf{Q}_{N,ff}(z) \end{pmatrix}$$

of the $2N \times 2N$ block matrix

$$\mathbf{M}_N = \begin{pmatrix} 0 & \eta_N \Pi_{p,N} \\ \eta_N \Pi_{f,N} & 0 \end{pmatrix}.$$

and deduce the results on $Q_N(z)$ using the identity $\mathbf{Q}_{N,pp}(z) = zQ_N(z^2)$. If $t_N(z)$ represents the Stieltjes transform of the probability measure $\frac{1}{c_N}(\tilde{\nu}_N - (1-c_N)\delta_0)$ (i.e. $t_N(z)$ is defined by $c_N t_N(z) = \tilde{t}_N(z) + \frac{1-c_N}{z}$), we establish that, in a certain sense, $\mathbf{Q}_{N,pp}$ and $\mathbf{Q}_{N,ff}$ behave as $z\tilde{t}_N(z^2)I_N$ while $\mathbf{Q}_{N,pf}$ and $\mathbf{Q}_{N,fp}$ behave as $t_N(z^2)I_N$. These results allows to justify that $\hat{\nu}_N - \tilde{\nu}_N \to 0$ and, after some work, imply that for each $\epsilon > 0$, almost surely, for N large enough, all the eigenvalues strictly less than 1 of $\Pi_{p,N}\Pi_{f,N}$ belong to $[0, 4c_*(1-c_*) + \epsilon]$.

When the signal is present.

In the second part of Chapter 5, we finally assume that the useful signal u is present, and, of course do not assume that $R_N = I$. We formulate the same assumptions on the useful signal than in Chapter 4, except that we replace the hypotheses related to the convergence of the empirical eigenvalue distribution of R_N and of measure β_N defined by (1.12) by the mild assumption that the $r \times r$ matrix sequence $\Theta_N^* (I \otimes R_N)^{-1} \Theta_N$ converges towards a positive definite matrix denoted G_* in the following. After expressing the orthogonal projection matrices $\Pi_{i,N}$ on the row spaces of $\Sigma_{i,N}$ as a low rank perturbation of matrices $\Pi_{i,N}^W$ (we now denote by $\Pi_{i,N}^W$ the projection matrix on the row space of $W_{i,N}$), and using the above results related to the behaviour of the resolvent $\mathbf{Q}_N^W$ of $\Pi_{p,N}^W \Pi_{f,N}^W$, we eventually obtain a clear characterization of the eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ that escape from the interval $[0, 4c_*(1-c_*)]$. We establish that for N large enough, the number of eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ that escape from $[0, 4c_*(1-c_*)]$ coincides with the number s of solutions of the equation

$$\det\left(x\left(\frac{\tilde{t}_*(x)}{(1-c)t_*(x)}\right)^2 - F_*\right) = 0 \tag{1.15}$$

that are larger than $4c_*(1-c_*)$, where $\tilde{t}_*$ and t_* represent the limits of the Stieltjes transforms $\tilde{t}_N$ and t_N (i.e. $\tilde{t}_*(z)$ is obtained by replacing c_N by c_* in (1.14)) and where matrix F_* is the rank P matrix (because Ω_* is itself a rank P matrix) defined as

$$F_* = \left(I + \Delta_*^{-1} G_*^{-1} \Delta_*^{-1}\right)^{-1} \Omega_*^* \left(I + \Delta_*^{-1} G_*^{-1} \Delta_*^{-1}\right)^{-1} \Omega_* \tag{1.16}$$

Using the explicit expressions of $\tilde{t}_*(x)$ and $t_*(x)$, we verify that when x increases from $4c_*(1-c_*)$ to 1, then $x\left(\frac{\tilde{t}_*(x)}{(1-c)t_*(x)}\right)^2$ increases from $\frac{c_*}{1-c_*} < 1$ to 1 if $c_* < 1/2$ and from $\frac{c_*}{1-c_*} \geq 1$ to $\left(\frac{c_*}{1-c_*}\right)^2$ if $c_* \geq 1/2$. As matrix F_* has rank P and verifies $F_* < I$, we deduce immediately from this that $s = 0$ if $c_* \geq 1/2$. If $c_* < \frac{1}{2}$, $s \leq P$, and s coincides with the number of eigenvalues of matrix F_* that are larger than $\frac{c_*}{1-c_*}$. Finally, if $x_{1,*} \geq x_{2,*} \ldots \geq x_{s,*}$ are the solutions of (1.15) larger than $4c_*(1-c_*)$, then the s largest eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ converge towards $x_{1,*} \geq x_{2,*} \ldots \geq x_{s,*}$. If $c_* < \frac{1}{2}$, it turns out that, under the assumption that the smallest eigenvalue of F_* is strictly larger than $\frac{c_*}{1-c_*}$, s coincides with P, and that it is possible to estimate P consistently as the number of eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ that are larger than $4c_*(1-c_*)$. From the expression (1.16), this condition will intuitively hold if both the r eigenvalues of $R_{u,N}^L$ are large enough (thus making matrix Δ_*^{-1} small) and the canonical correlation coefficients between the past and the future of u large enough as well (thus making the singular values of Ω_* large).

Chapitre 2

Some notations and basic tools.

In this chapter we introduce the assumptions and notations which will be used throughout the manuscript as well as some fundamental tools.

Assumptions

— We assume that L is a fixed parameter, and that M and N converge towards $+\infty$ in such a way that

$$c_N = \frac{ML}{N} \to c_*,\ c_* > 0. \tag{2.1}$$

This regime will be referred to as $N \to +\infty$ in the following. In the regime (2.1), M should be interpreted as an integer $M = M(N)$ depending on N. The various matrices we have introduced above thus depend on N and will be denoted $R_N, Y_{f,N}, Y_{p,N}, \ldots$. In order to simplify the notations, the dependency w.r.t. N will sometimes be omitted.

— The sequence of covariance matrices $(R_N)_{N\geq 1}$ of M–dimensional vectors $(v_n)_{n=1,\ldots,N}$ is supposed to verify

$$a\, I \leq R_N \leq b\, I \tag{2.2}$$

for each N, where $a > 0$ and $b > 0$ are 2 constants. $\lambda_{1,N} \geq \lambda_{2,N} \geq \ldots \geq \lambda_{M,N}$ represent the eigenvalues of R_N arranged in the decreasing order and $f_{1,N}, \ldots, f_{M,N}$ denote the corresponding eigenvectors. Hypothesis (2.2) is obviously equivalent to $\lambda_{M,N} \geq a$ and $\lambda_{1,N} \leq b$ for each N.

Notations

— For each $1 \leq i \leq 2L$ and $1 \leq m \leq M$, $\mathbf{f}_i^m$ represents the vector of the canonical basis of $\mathbb{C}^{2ML}$ with 1 at the index $m + (i-1)M$ and zeros elsewhere. In order to simplify the notations, we mention that if $i \leq L$, vector $\mathbf{f}_i^m$ may also represent the vector of the canonical basis of $\mathbb{C}^{ML}$ with 1 at the index $m + (i-1)M$ and zeros elsewhere. Vector $\mathbf{e}_j$ with $1 \leq j \leq N$ represents the j –th vector of the canonical basis of $\mathbb{C}^N$.

— For each integers $l \in \mathbb{Z}$ and $K \in \mathbb{N}$ such that $K \geq |l|$, we define $K \times K$ "shift" matrix $J_K^{(l)}$ as

$$(J_K^{(l)})_{ij} = \delta_{j-i,l}. \tag{2.3}$$

— $\mathbb{R}^+$ and $\mathbb{R}^-$ represents respectively the set of all non-negative numbers and non-positive numbers, and we denote $\mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$, $\mathbb{R}^{+*} \equiv \mathbb{R}^+ \setminus \{0\}$ and $\mathbb{R}^{-*} \equiv \mathbb{R}^- \setminus \{0\}$. We also define $\mathbb{C}^+ \equiv z \in \mathbb{C} : \mathrm{Im}(z) > 0$

— By a nice constant, we mean a positive deterministic constant which does not depend on the dimensions M and N nor of the complex variable z. In the following, κ will represent a generic nice constant whose value may change from one line to the other. A nice polynomial $P(z)$ is a polynomial whose degree and coefficients are nice constants.

— If $(\alpha_N)_{N\geq 1}$ is a sequence of positive real numbers and if Ω is a domain of $\mathbb{C}$, we will say that a sequence of functions $(f_N(z))_{N\geq 1}$ verifies $f_N(z) = \mathcal{O}_z(\alpha_N)$ for $z \in \Omega$ if there exists two nice polynomials P_1 and P_2 such that $|f_N(z)| \leq \alpha_N P_1(|z|) P_2(\frac{1}{|\mathrm{Im} z|})$ for each $z \in \Omega$. If $\Omega = \mathbb{C}^+$, we will just write $f_N(z) = \mathcal{O}_z(\alpha_N)$ without mentioning the domain. We notice that if P_1, P_2 and Q_1, Q_2 are nice polynomials, then $P_1(|z|)P_2(\frac{1}{|\mathrm{Im} z|}) + Q_1(|z|)Q_2(\frac{1}{|\mathrm{Im} z|}) \leq (P_1 + Q_1)(|z|)(P_2 + Q_2)(\frac{1}{|\mathrm{Im} z|})$, from which we conclude that if the sequences $(f_{1,N})_{N\geq 1}$ and $(f_{2,N})_{N\geq 1}$ are $\mathcal{O}_z(\alpha_N)$ on Ω, then it also holds $f_{1,N}(z) + f_{2,N}(z) = \mathcal{O}_z(\alpha_N)$ on Ω.

— For any matrix A, $\|A\|$ and $\|A\|_F$ represent its spectral norm and Frobenius norm respectively. The transpose, conjugate, and conjugate transpose of A are respectively denoted by A^T ,$\bar{A}$ and A^*, for matrix B of the same size $A \geq B$ stands for $A - B$ non-negative definite. If moreover A is a square matrix, $\mathrm{Im}(A)$ is the Hermitian matrix defined by $\mathrm{Im}(A) = \frac{A-A^*}{2i}$.

— $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ represents the set of all $\mathcal{C}^\infty$ real valued compactly supported functions defined on $\mathbb{R}$.

— If ξ is a random variable, we denote by ξ° the zero mean random variable defined by

$$\xi^\circ = \xi - \mathbb{E}\xi. \tag{2.4}$$

Fundamentals tools

We remind here one of the basic tool in random matrix theory, i.e. the Stieltjes transform.

Let μ be the finite measure with a support $\mathrm{Supp}(\mu) \in \mathbb{R}$, then its Stieltjes transform f_μ is defined as

$$f_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \quad \text{for each } z \in \mathbb{C} \setminus \mathrm{Supp}(\mu)$$

We first state well known properties of Stieltjes transforms (see e.g. the Appendix of [27], the Appendix A of [18], and the references therein).

Proposition 2.1. *The following properties hold true :*
1. Let f be the Stieltjes transform of a positive finite measure μ, then
– the function f is analytic over $\mathbb{C}^+$,
– if $z \in \mathbb{C}^+$ then $f(z) \in \mathbb{C}^+$,
– the function f satisfies : $|f(z)| \leq \frac{\mu(\mathbb{R})}{\mathrm{Im} z}$, for $z \in \mathbb{C}^+$
– if $\mu(-\infty, 0) = 0$ then its Stieltjes transform f is analytic over $\mathbb{C}/\mathbb{R}^+$. Moreover, $z \in \mathbb{C}^+$ implies $zf(z) \in \mathbb{C}^+$.
– for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ we have

$$\int_{\mathbb{R}} \phi(\lambda) d\mu(\lambda) = \frac{1}{\pi} \lim_{y \downarrow 0} \mathrm{Im} \left\{ \int_{\mathbb{R}} \phi(x) f(x + iy) dx \right\}.$$

2. Conversely, let f be a function analytic over $\mathbb{C}^+$ such that $f(z) \in \mathbb{C}^+$ if $z \in \mathbb{C}^+$ and for which $\sup_{y \geq \epsilon} |iyf(iy)| < +\infty$ for some $\epsilon > 0$. Then, f is the Stieltjes transform of a unique positive finite measure μ such that $\mu(\mathbb{R}) = \lim_{y \to +\infty} -iyf(iy)$. If moreover $zf(z) \in \mathbb{C}^+$ for z in $\mathbb{C}^+$ then, $\mu(\mathbb{R}^-) = 0$. In particular, f is given by

$$f(z) = \int_0^{+\infty} \frac{\mu(d\lambda)}{\lambda - z}$$

and has an analytic continuation on $\mathbb{C}/\mathbb{R}^+$.
3. Let F be an $P \times P$ matrix-valued function analytic on $\mathbb{C}^+$ verifying
– $\mathrm{Im}(F(z)) > 0$ if $z \in \mathbb{C}^+$
– $\sup_{y > \epsilon} \|iyF(iy)\| < +\infty$ for some $\epsilon > 0$.
Then, $F \in \mathcal{S}_P(\mathbb{R})$, and if μ^F is the corresponding $P \times P$ associated positive measure, it holds that

$$\mu^F(\mathbb{R}) = \lim_{y \to +\infty} -iyF(iy). \tag{2.5}$$

If moreover $\mathrm{Im}(zF(z)) > 0$, then, $F \in \mathcal{S}_P(\mathbb{R}^+)$.

One of the classical approach of random matrix theory is based on the fact that for the ensemble of random hermitian $N \times N$ matrices $\mathcal{M}_N$, the Stieltjes transform of its spectral distribution $F^{\mathcal{M}}(\lambda)$ which is defined as

$$F(\lambda) = \frac{1}{N} Card\{\text{eigenvalues of } \mathcal{M}_N \leq \lambda\}$$

coincides to the $\frac{1}{N}\mathrm{Tr} Q_N(z)$, where $Q_N(z)$ is the resolvent of matrix $\mathcal{M}_N$ which is defined by

$$Q_N(z) = (\mathcal{M}_N - z)^{-1}$$

It is well known that resolvent of hermitian matrix is bounded for $z \in \mathbb{C}^+$ and from classical linear algebra we have so called resolvent identity, more precisely :

— $\|Q_N(z)\| \leq (\mathrm{Im} z)^{-1}$, for each $z \in \mathbb{C}^+$

— *The resolvent identity :* $zQ_N(z) = -I_N + Q_N(z)\mathcal{M}_N$

these two properties will be used a lot throughout the manuscript.

We finally recall the 2 Gaussian tools that will be used in the sequel in order to evaluate the asymptotic behaviour of corresponding resolvent. The corresponding proofs can be found for example in [39] and [11].

Proposition 2.2. ***(Integration by parts formula.)*** *Let $\xi = [\xi_1, \ldots, \xi_K]^T$ be a complex Gaussian random vector such that $\mathbb{E}\{\xi\} = 0$, $\mathbb{E}\{\xi\xi^T\} = 0$ and $\mathbb{E}\{\xi\xi^*\} = \Omega$. If $\Gamma : (\xi) \mapsto \Gamma(\xi, \bar{\xi})$ is a $\mathcal{C}^1$ complex function polynomially bounded together with its derivatives, then*

$$\mathbb{E}\{\xi_i \Gamma(\xi)\} = \sum_{k=1}^{K} \Omega_{ik} \mathbb{E}\left\{\frac{\partial \Gamma(\xi)}{\partial \bar{\xi}_k}\right\}. \tag{2.6}$$

Proposition 2.3. ***(Poincaré-Nash inequality.)*** *Let $\xi = [\xi_1, \ldots, \xi_K]^T$ be a complex Gaussian random vector such that $\mathbb{E}\{\xi\} = 0$, $\mathbb{E}\{\xi\xi^T\} = 0$ and $\mathbb{E}\{\xi\xi^*\} = \Omega$. If $\Gamma : (\xi) \mapsto \Gamma(\xi, \bar{\xi})$ is a $\mathcal{C}^1$ complex function polynomially bounded together with its derivatives, then, noting $\nabla_\xi \Gamma = [\frac{\partial \Gamma}{\partial \xi_1}, \ldots, \frac{\partial \Gamma}{\partial \xi_K}]^T$ and $\nabla_{\bar{\xi}} \Gamma = [\frac{\partial \Gamma}{\partial \bar{\xi}_1}, \ldots, \frac{\partial \Gamma}{\partial \bar{\xi}_K}]^T$*

$$\mathbf{Var}\{\Gamma(\xi)\} \leq \mathbb{E}\left\{\nabla_\xi \Gamma(\xi)^T \Omega \overline{\nabla_\xi \Gamma(\xi)}\right\} + \mathbb{E}\left\{\nabla_{\bar{\xi}} \Gamma(\xi)^* \Omega \nabla_{\bar{\xi}} \Gamma(\xi)\right\}. \tag{2.7}$$

Chapitre 3

Large empirical autocovariance matrices between the past and the future

This chapter is dedicated to the study of the singular values of matrix $\hat{R}^L_{f|p,y}$, or equivalently of the eigenvalues of $\hat{R}^L_{f|p,y}(\hat{R}^L_{f|p,y})^*$, in the case when the signal is absent, i.e. $y_n = v_n$. In this context, it thus holds that $\frac{1}{\sqrt{N}}Y_{p,N} = \frac{1}{\sqrt{N}}V_{p,N}$ and $\frac{1}{\sqrt{N}}Y_{f,N} = \frac{1}{\sqrt{N}}V_{f,N}$. In the following, we denote by $W_{p,N}$ and $W_{f,N}$ the normalized matrices

$$W_{p,N} = \frac{1}{\sqrt{N}}V_{p,N},\ W_{f,N} = \frac{1}{\sqrt{N}}V_{f,N}.$$

We recall that the resolvent $Q_N(z)$ of $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ is defined by

$$Q_N(z) = \left(W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^* - zI\right)^{-1}. \tag{3.1}$$

As the direct study of $Q_N(z)$ is not obvious, we rather introduce the resolvent $\mathbf{Q}_N(z)$ of the $2ML \times 2ML$ block matrix

$$\mathbf{M}_N = \begin{pmatrix} 0 & W_{f,N}W_{p,N}^* \\ W_{p,N}W_{f,N}^* & 0 \end{pmatrix}. \tag{3.2}$$

It is well known that $\mathbf{Q}_N(z)$ can be expressed as

$$\mathbf{Q}_N(z) = \begin{pmatrix} zQ_N(z^2) & Q_N(z^2)W_{f,N}W_{p,N}^* \\ W_{p,N}W_{f,N}^*Q_N(z^2) & z\tilde{Q}_N(z^2) \end{pmatrix}, \tag{3.3}$$

where $\tilde{Q}_N(z)$ is the resolvent of matrix $W_{p,N}W_{f,N}^*W_{f,N}W_{p,N}^*$. As shown below, it is rather easy to evaluate the asymptotic behaviour of $\mathbf{Q}_N(z)$ using the Poincaré-Nash inequality and the integration by part formula (see Propositions 2.3 and 2.2 below). Formula (3.3) will then provide all the necessary information on $Q_N(z)$.

In the following, every $2ML \times 2ML$ matrix $\mathbf{G}$ will be written as

$$\mathbf{G} = \begin{pmatrix} \mathbf{G_{pp}} & \mathbf{G_{pf}} \\ \mathbf{G_{fp}} & \mathbf{G_{ff}} \end{pmatrix},$$

where the 4 matrices $(\mathbf{G_{ij}})_{i,j\in p,f}$ are $ML \times ML$. Sometimes, the blocks will be denoted $\mathbf{G}(pp)$, $\mathbf{G}(pf)$, ...

We denote by W_N the $2ML \times N$ matrix defined by

$$W_N = \begin{pmatrix} W_{p,N} \\ W_{f,N} \end{pmatrix}. \tag{3.4}$$

Its elements $(W^m_{i,j})_{i\le 2L, j\le N, m\le M}$ satisfy

$$\mathbb{E}\{W^m_{i,j}(W^{m'}_{i',j'})^*\} = \frac{1}{N}R_{mm',N}\delta_{i+j,i'+j'},$$

where $W_{i,j}^m$ represents the element which lies on the $(m + M(i-1))$-th line and j-th column for $1 \leq m \leq M$, $1 \leq i \leq 2L$ and $1 \leq j \leq N$. Similarly, $\mathbf{Q}_{i_1 i_2}^{m_1 m_2}$, where $1 \leq m_1, m_2 \leq M$ and $1 \leq i_1, i_2 \leq 2L$, represents the entry $(m_1 + M(i_1 - 1)), (m_2 + M(i_2 - 1))$ of $\mathbf{Q}$. For each $j = 1, \ldots, N$, $\{w_j\}_{j=1}^N, \{w_{p,j}\}_{j=1}^N$ and $\{w_{f,j}\}_{j=1}^N$ are the column of matrices W, W_p and W_f respectively.

3.1 On the literature.

The large sample behaviour of high-dimensional autocovariance matrices was comparatively less studied than the high-dimensional covariance matrices. We first mention [26] which studied the asymptotic behaviour of the eigenvalue distribution of the hermitian matrix $\hat{R}_\tau + \hat{R}_\tau^*$ where $\hat{R}_\tau$ is defined as $\hat{R}_\tau = \frac{1}{N}\sum_{n=1}^N x_{n+\tau} x_n^*$ where $(x_n)_{n\in\mathbb{Z}}$ represents a M dimensional non Gaussian i.i.d. sequence, the components of each vector x_n being moreover i.i.d. with zero means and unit variance. In particular, $\mathbb{E}(x_n x_n^*) = I$. It is proved that the empirical eigenvalue distribution of $\hat{R}_\tau + \hat{R}_\tau^*$ converges towards a limit distribution independent from $\tau \geq 1$. Using finite rank perturbation technics of the resolvent of the matrix under consideration, the Stieltjes transform of this distribution was shown to satisfy a polynomial degree 3 equation. Solving this equation led to an explicit expression of the probability density of the limit distribution. [31] extended these results to the case where $(x_n)_{n\in\mathbb{Z}}$ is a non Gaussian linear process $x_n = \sum_{l=0}^{+\infty} A_l z_{n-l}$ where $(z_n)_{n\in\mathbb{Z}}$ is i.i.d., and where matrices $(A_l)_{l\geq 0}$ are simultaneously diagonalizable. The limit eigenvalue distribution was characterized through its Stieltjes transform that is obtained by integration of a certain kernel, itself solution of an integral equation. The proof was based on the observation that in the Gaussian case, the correlated vectors $(x_n)_{n\in\mathbb{Z}}$ can be replaced by independent ones using a classical frequency domain decorrelation procedure. The results were generalized in the non Gaussian case using the generalized Lindeberg principle. We also mention [2] (see also the book [3]) where the existence of a limit distribution of any symmetric polynomial of $(\hat{R}_\tau, \hat{R}_\tau^*)_{\tau\in T}$ for some finite set T was proved using the moment method when x is a linear non Gaussian process. [28] studied the asymptotic behaviour of matrix $\hat{R}_\tau \hat{R}_\tau^*$ when $(x_n)_{n\in\mathbb{Z}}$ represents a M dimensional non Gaussian i.i.d. sequence, the components of each vector x_n being moreover i.i.d. Using finite rank perturbation technics, it was shown that the empirical eigenvalue distribution converges towards a limit distribution whose Stieltjes transform is solution of a degree 3 polynomial equation. As in [26], this allowed to obtain the expression of the corresponding probability density function. Using combinatorial technics, [28] also established that almost surely, for large enough dimensions, all the eigenvalues of $\hat{R}_\tau \hat{R}_\tau^*$ are located in a neighbourhood of the support of the limit eigenvalue distribution. We finally mention that [29] used the results in [28] in order to study the largest eigenvalues and corresponding eigenvectors of $\hat{R}_\tau \hat{R}_\tau^*$ when the observation contains a certain spiked useful signal that is more specific than the signals $(u_n)_{n\in\mathbb{Z}}$ that motivated this thesis.

We now compare the results of the present chapter with the content of the above previous works. We first study a matrix that is more general than $\hat{R}_\tau \hat{R}_\tau^*$. While we do not consider linear processes here, we do not assume that the covariance matrix of the i.i.d. sequence $(v_n)_{n\in\mathbb{Z}}$ is reduced to I as in [28]. This in particular implies that the Stieltjes transform of the deterministic equivalent ν_N of $\hat{\nu}_N$ cannot be evaluated in closed from. Therefore, a dedicated analysis of the support and of the properties of ν_N is provided here. We also mention that in contrast with the above papers, we characterize the asymptotic behaviour of the resolvent of matrix $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ while the mentionned previous works only studied the normalized trace of the resolvent of the matrices under consideration. Studying the full resolvent matrix is necessary to address the case where a useful spiked signal u is added to the noise v. We notice that the above papers addressed the non Gaussian case while we consider the case where v is a complex Gaussian i.i.d. sequence. This situation is of course simpler in that various Gaussian tools are available, but appears to be relevant because in the context of the present thesis, v is indeed supposed to represent some additive noise, which, in a number of contexts, is Gaussian.

We finally mention that some of the results may be obtained by adapting general recent results devoted to the study of the spectrum of hermitian polynomials of GUE matrices and deterministic matrices (see [6] and [34]). If we denote by Z_N the $M \times (N + 2L - 1)$ matrix $Z_N = (v_1, \ldots, v_{N+2L-1})$, then Z_N can be written as $Z_N = R_N^{1/2} X_N$ where the entries of X_N are i.i.d. complex Gaussian standard variables. Each $M \times M$ block $\Sigma_{N,k,l}$ $(1 \leq k, l \leq L)$ of $\Sigma_N = W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ is clearly a polynomial of X_N, X_N^* and various

$M \times M$ and $M \times (N+2L-1)$ deterministic matrices. Assume that $M < N+2L-1$. In order to be back to a polynomial of GUE matrices, it is possible to consider the $L(N+2L-1) \times L(N+2L-1)$ matrix $\tilde{\Sigma}_N$ whose $(N+2L-1) \times (N+2L-1)$ blocks are defined by

$$\tilde{\Sigma}_{N,k,l} = \begin{pmatrix} \Sigma_{N,k,l} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that apart 0, the eigenvalues of $\tilde{\Sigma}_N$ coincide with those of Σ_N. If $\tilde{X}_N$ is any $(N+2L-1)\times(N+2L-1)$ matrix with i.i.d. complex Gaussian standard entries whose M first rows coincide with X_N, then, it is easily seen that each block of $\tilde{\Sigma}_N$ coincides with a hermitian polynomial of $\tilde{X}_N, \tilde{X}_N^*$ and deterministic $(N+2L-1) \times (N+2L-1)$ matrices such as

$$\tilde{R}_N = \begin{pmatrix} R_N & 0 \\ 0 & 0 \end{pmatrix}.$$

Expressing $\tilde{X}_N$ as the sum of its hermitian and anti-hermitian parts, we are back to study the behaviour of the eigenvalues of a matrix whose blocks are hermitian polynomials of 2 independent GUE matrices and of $(N+2L-1) \times (N+2L-1)$ deterministics matrices. Extending Proposition 2.2 and Theorem 1.1 in [6] to block matrices (as in Corollary 2.3 in [34]) would lead to the conclusion that $\hat{\nu}_N$ has a deterministic equivalent ν_N and that the eigenvalues of $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ are located in the neighbourhood of the support of ν_N. While this last consequence would avoid the use of the specific approach used in section 3.7 of the present chapter, the existence of ν_N is not a sufficient information. ν_N should of course be characterized through its Stieltjes transform, and we believe that the adaptation of Proposition 2.2 and Theorem 1.1 in [6] is not the most efficient approach.

3.2 Use of the Poincaré-Nash inequality.

In this paragraph, we control the variance of various functionals of $\mathbf{Q}_N(z)$ using the Poincaré-Nash inequality. For this, it appears useful to evaluate the moments of $\|W_N\|$. The following result holds.

Lemma 3.1. *For any $l \in \mathbb{N}$, it holds that* $\sup_{N \geq 1} \mathbb{E}\{\|W_N\|^{2l}\} < +\infty$.

Proof. We first remark that it is possible to be back to the case where matrix $R_N = I_M$. Due to the Gaussianity of the i.i.d. vectors $(v_n)_{n\geq 1}$, it exists i.i.d. $\mathcal{N}_c(0, I_M)$ distributed vectors $(v_{iid,n})_{n\geq 1}$ such that $\mathbb{E}(v_{iid,n}v_{iid,n}^*) = I_M$ verifying $v_n = R_N^{1/2} v_{iid,n}$. From this, we obtain immediately that the $2ML \times N$ block Hankel matrix $W_{iid,N}$ built from $(v_{n,iid})_{n=1,\ldots,N}$ satisfies

$$W_N = \begin{pmatrix} R_N^{1/2} & & \\ & \ddots & \\ & & R_N^{1/2} \end{pmatrix} W_{iid,N}. \tag{3.5}$$

As the spectral norm of R_N is assumed uniformly bounded when N increases, the statement of the lemma is equivalent to $\sup_N \mathbb{E}\{\|W_{iid}\|^{2l}\} < +\infty$. It is shown in [32] that the empirical eigenvalue distribution of $W_{iid,N}W_{iid,N}^*$ converges towards the Marcenko-Pastur distribution, and that its smallest non zero eigenvalue and its largest eigenvalue (which coincides with $\|W_{iid,N}\|^2$) converge almost surely towards $(1-\sqrt{c_*})^2$ and $(1+\sqrt{c_*})^2$ respectively. We express $\mathbb{E}\{\|W_{iid}\|^{2l}\}$ as

$$\begin{aligned}
\mathbb{E}\{\|W_{iid}\|^{2l}\} &= \mathbb{E}\{\|W_{iid}\|^{2l}\mathbf{1}_{\|W_{iid}\|^2 \leq (1+\sqrt{c_*})^2+\delta}\} + \mathbb{E}\{\|W_{iid}\|^{2l}\mathbf{1}_{\|W_{iid}\|^2 > (1+\sqrt{c_*})^2+\delta}\} \\
&\leq \kappa + \mathbb{E}\{\|W_{iid}\|_F^{2l}\mathbf{1}_{\|W_{iid}\|^2 > (1+\sqrt{c_*})^2+\delta}\} \leq \kappa + \mathbb{E}\{\|W_{iid}\|_F^{4l}\}^{1/2}\mathbb{E}\{\mathbf{1}_{\|W_{iid}\|^2 > (1+\sqrt{c_*})^2+\delta}\}^{1/2}
\end{aligned}$$

where $\kappa > 0$ is a nice constant. As $\mathbb{E}\{\|W_{i.i.d.}\|_F^{4l}\} = \mathcal{O}(N^{2l})$, it is sufficient to prove that $\mathbb{E}\{\mathbf{1}_{\|W_{iid}\|^2 > (1+\sqrt{c_*})^2+\delta}\}$ is less than any power of N^{-1}. We introduce a smooth function ϕ_0 defined on $\mathbb{R}$ by

$$\phi_0(\lambda) = \begin{cases} 1, \text{ for } \lambda \in [-\infty, -\delta] \cup [(1+\sqrt{c_*})^2 + \delta, +\infty], \\ 0, \text{ for } \lambda \in [-\delta/2, (1+\sqrt{c_*})^2 + \delta/2] \end{cases}$$

and $\phi_0(\lambda) \in (0, 1)$ elsewhere. Then, it holds that

$$\mathbb{E}\{\mathbf{1}_{\|W_{iid}\|^2 > (1+\sqrt{c_*})^2+\delta}\} = \mathbb{E}\{\mathbf{1}_{\lambda_{max}(W_{iid}W_{iid}^*) > (1+\sqrt{c_*})^2+\delta}\} \leq \mathbf{P}[\mathrm{Tr}\phi_0(W_{iid}W_{iid}^*) \geq 1]$$
$$\leq \mathbb{E}\{(\mathrm{Tr}\phi_0(W_{iid}W_{iid}^*))^{2k}\}$$

for any $k \in \mathbb{N}$. Lemma 3.1 thus appears as an immediate consequence of the following lemma.

Lemma 3.2. *For each smooth function ϕ such that $\phi(\lambda) = 0$ if $\lambda \in [-\delta/2, (1+\sqrt{c_*})^2 + \delta/2]$ and $\phi(\lambda)$ constant on $[-\infty, -\delta] \cup [(1+\sqrt{c_*})^2 + \delta, +\infty]$, it holds that $\forall k \in \mathbb{N}$, $\mathbb{E}\left\{(\mathrm{Tr}\phi(W_{iid}W_{iid}^*))^{2k}\right\} \leq \frac{\kappa}{N^{2k}}$.*

Proof. We prove the Lemma by induction. We first consider the case $k = 1$. For more convenience we will write W instead of W_{iid} in the course of the proof. Here and below we take sum for all possible values of indexes, if not specified. From (2.7) we have

$$\mathbf{Var}\{\mathrm{Tr}\phi(WW^*)\} \leq \sum \mathbb{E}\left\{\left(\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial \overline{W}_{i_1,j_1}^{m_1}}\right)^* \mathbb{E}\{W_{i_1,j_1}^{m_1}\overline{W}_{i_2,j_2}^{m_2}\} \frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial \overline{W}_{i_2,j_2}^{m_2}}\right\}$$
$$+ \sum \mathbb{E}\left\{\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial W_{i_1,j_1}^{m_1}} \mathbb{E}\{W_{i_1,j_1}^{m_1}\overline{W}_{i_2,j_2}^{m_2}\} \left(\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial W_{i_2,j_2}^{m_2}}\right)^*\right\}. \tag{3.6}$$

We only evaluate the first term, denoted by ψ, of the right handside of (3.6), because the second one can be addressed similarly. For this, we first remark that

$$\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial \overline{W}_{i_1,j_1}^{m_1}} = \mathrm{Tr}\left(\phi'(WW^*)\frac{\partial WW^*}{\partial \overline{W}_{i_1,j_1}^{m_1}}\right) = \left(\phi'(WW^*)W\right)_{i_1,j_1}^{m_1}.$$

Plugging this into (3.6) we obtain

$$\psi = \sum \frac{1}{N}\mathbb{E}\left\{\left(\phi'(WW^*)W\right)_{j_1,i_1}^{*m_1} \delta_{m_1,m_2}\delta_{i_1+j_1,i_2+j_2} \left(\phi'(WW^*)W\right)_{i_2,j_2}^{m_2}\right\}.$$

Denoting $l = i_1 - i_2$, it is easy to verify that ψ can be written as

$$\psi = \frac{1}{N}\sum_{l=-(L-1)}^{L-1} \mathbb{E}\{\mathrm{Tr}\left(\phi'(WW^*)W\right)^* (J_L^{(l)} \otimes I_M)\left(\phi'(WW^*)W\right) J_N^{(l)}\}. \tag{3.7}$$

where we recall that matrix J_L is defined by (2.3). For each $ML \times N$ matrices A and B, the Schwartz inequality and the inequality between arithmetic and geometric means lead to

$$\left|\frac{1}{N}\mathrm{Tr}A^*(J_L^{*\epsilon(u)} \otimes I_M)BJ_N^{*\epsilon(u)}\right| \leq \frac{1}{2N}\mathrm{Tr}A^*(J_L^{*\epsilon(u)}J_L^{(u)} \otimes I_M)A + \frac{1}{2N}\mathrm{Tr}B^*J_N^{*\epsilon(u)}J_N^{(u)}B.$$

Therefore, since $J_L^{*\epsilon(u)}J_L^{(u)} \otimes I_M \leq I_{ML}$ and $J_N^{*\epsilon(u)}J_N^{(u)} \leq I_N$

$$\left|\frac{1}{ML}\mathrm{Tr}A^*(J_L^{*\epsilon(u)} \otimes I_M)BJ_N^{*\epsilon(u)}\right| \leq \frac{\kappa}{N}(\mathrm{Tr}A^*A + \mathrm{Tr}B^*B). \tag{3.8}$$

By taking here $A = B = \phi'(WW^*)W$, we obtain from (3.6) and (3.7)

$$\mathbf{Var}\{\mathrm{Tr}\phi(WW^*)\} \leq \frac{\kappa}{N}\mathbb{E}\left\{\mathrm{Tr}\left(\phi'(WW^*)\right)^2 WW^*\right\}. \tag{3.9}$$

Consider the function $\eta(\lambda) = (\phi'(\lambda))^2\lambda$. It is clear that $\eta(\lambda)$ is a compactly supported smooth function. Therefore (see e.g. [32]), it holds that

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\left((\phi'(WW^*))^2 WW^*\right)\right\} = \int_{\mathcal{S}_{MP,N}} \eta(\lambda)d\mu_{MP,N}(\lambda) + \mathcal{O}\left(\frac{1}{N^2}\right),$$

where $\mu_{MP,N}$ is the measure associated to Marcenko-Pastur distribution with parameters $(1, c_N)$ and where $\mathcal{S}_{MP,N} \subset [0, (1+\sqrt{c_N})^2]$ represents the support of $\mu_{MP,N}$. It is clear that for N large enough, the support of ϕ' and $\mathcal{S}_{MP,N}$ do not intersect, so that $\int_{\mathcal{S}_{MP,N}} \eta(\lambda) d\mu_{MP,N}(\lambda) = 0$. Therefore, we obtain that

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\left((\phi'(WW^*))^2 WW^*\right)\right\} = \mathcal{O}\left(\frac{1}{N^2}\right).$$

This and (3.9) lead to the conclusion that $\mathbf{Var}\{\mathrm{Tr}\phi(WW^*)\} = \mathcal{O}\left(N^{-2}\right)$. To finalize the case $k=1$, we express $\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^2\}$ as $\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^2\} = \mathbf{Var}\{\mathrm{Tr}\phi(WW^*)\} + \mathbb{E}\{\mathrm{Tr}\phi(WW^*)\}^2$. [32, Lemma 10.1] implies that $\mathbb{E}\{\mathrm{Tr}\phi(WW^*)\} = \mathcal{O}(N^{-1})$, which completes the proof for $k=1$.
Now we suppose that for any $n \leq k$ we have $\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2n}\} = \mathcal{O}(N^{-2n})$ and are about to prove that it holds for $n = k+1$. As in the previous case we write

$$\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2(k+1)}\} = \mathbf{Var}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\} + \left(\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\}\right)^2. \tag{3.10}$$

To evaluate the second term of the r.h.s. of (3.10), we use the Schwartz inequality and the induction assumption

$$\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\} \leq \left(\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2k}\}\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^2\}\right)^{1/2} = \mathcal{O}\left(\frac{1}{N^{k+1}}\right), \tag{3.11}$$

We follow the same steps as in the case $k=1$ to study the first term of the r.h.s. of (3.10). Using again the Poincaré-Nash inequality, we obtain that

$$\mathbf{Var}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\} \leq \frac{\kappa}{N}\mathbb{E}\left\{(\mathrm{Tr}\phi(WW^*))^{2k}\,\mathrm{Tr}\left(\phi'(WW^*)^2 WW^*\right)\right\}.$$

Using Holder's inequality, we obtain

$$\mathbf{Var}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\} \leq \frac{\kappa}{N}\mathbb{E}\left\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\right\}^{\frac{k}{k+1}}\mathbb{E}\left\{\left(\mathrm{Tr}(\phi'(WW^*)^2 WW^*)\right)^{k+1}\right\}^{\frac{1}{k+1}}. \tag{3.12}$$

The properties of function $\eta(\lambda) = \phi'(\lambda)^2\lambda$ imply that it satisfies the induction hypothesis and that it verifies (3.11), i.e. $\mathbb{E}\left\{(\mathrm{Tr}(\phi'(WW^*)^2 WW^*))^{k+1}\right\} = \mathcal{O}(\frac{1}{N^{k+1}})$. Plugging this into (3.12), we get that

$$\mathbf{Var}\{(\mathrm{Tr}\phi(WW^*))^{k+1}\} \leq \frac{\kappa}{N^2}\mathbb{E}\left\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\right\}^{\frac{k}{k+1}}.$$

From this, (3.11) and (3.10), we immediately obtain

$$\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\} \leq \frac{\kappa_1}{N^2}\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\}^{\frac{k}{k+1}} + \frac{\kappa_2}{N^{2k+2}}. \tag{3.13}$$

We denote by $z_{k,N}$ the term $z_{k,N} = N^{2k+2}\,\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\}$. Then, (3.13) implies that

$$z_{k,N} \leq \kappa_1 \left(z_{k,N}\right)^{k/(k+1)} + \kappa_2.$$

This inequality leads to the conclusion that sequence $(z_{k,N})_{N\geq 1}$ is bounded, or equivalently that $\mathbb{E}\{(\mathrm{Tr}\phi(WW^*))^{2k+2}\} \leq \frac{\kappa}{N^{2k+2}}$ as expected. This completes the proof of Lemmas 3.2 and 3.1. ■

We now evaluate the variance of useful functionals of the resolvent $\mathbf{Q}_N(z)$.

Lemma 3.3. *Let $(F_N)_{N\geq 1}$, $(G_N)_{N\geq 1}$ be sequences of deterministic $2ML \times 2ML$ matrices and $(H_N)_{N\geq 1}$ a sequence of deterministic $N \times N$ matrices such that* $\max\{\sup_N \|F_N\|, \sup_N \|G_N\|, \sup_N \|H_N\|\} \leq \kappa$. *Then, for each $z \in \mathbb{C}^+$, it holds that*

$$\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}F\mathbf{Q}\right\} \leq \frac{C(z)\kappa^2}{N^2}, \tag{3.14}$$

$$\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}F\mathbf{Q}GWHW^*\right\} \leq \frac{C(z)\kappa^6}{N^2}. \tag{3.15}$$

where $C(z)$ can be written as $C(z) = P_1(|z|)P_2\left(\frac{1}{\mathrm{Im}z}\right)$ for some nice polynomials P_1 and P_2.

Proof. We first prove (3.14) and denote by ξ the term $\xi = \frac{1}{ML}\mathrm{Tr}F\mathbf{Q}$. The Poincare-Nash inequality leads to

$$\mathbf{Var}\{\xi\} \leq \sum_{\substack{i_1,j_1,m_1\\i_2,j_2,m_2}} \mathbb{E}\left\{ \left(\frac{\partial \xi}{\partial \overline{W}^{m_1}_{i_1,j_1}} \right)^* \mathbb{E}\{W^{m_1}_{i_1,j_1} \overline{W}^{m_2}_{i_2,j_2}\} \frac{\partial \xi}{\partial \overline{W}^{m_2}_{i_2,j_2}} \right\}$$
$$+ \sum_{\substack{i_1,j_1,m_1\\i_2,j_2,m_2}} \mathbb{E}\left\{ \frac{\partial \xi}{\partial W^{m_1}_{i_1,j_1}} \mathbb{E}\{W^{m_1}_{i_1,j_1} \overline{W}^{m_2}_{i_2,j_2}\} \left(\frac{\partial \xi}{\partial W^{m_2}_{i_2,j_2}} \right)^* \right\}.$$

We just evaluate the first term of the r.h.s. that we denote by ϕ. For this, we need the expression of the derivative of $\mathbf{Q}$ with respect to the complex conjugates of the entries of W. We denote by Π_{pf} and Π_{fp} the $2ML \times 2ML$ matrices defined by $\Pi_{pf} = \left(\begin{smallmatrix} 0 & I_{ML} \\ 0 & 0 \end{smallmatrix}\right)$ and $\Pi_{fp} = \left(\begin{smallmatrix} 0 & 0 \\ I_{ML} & 0 \end{smallmatrix}\right)$. Then, after some algebra, we obtain that

$$\begin{aligned}\frac{\partial \mathbf{Q}}{\partial \overline{W}^m_{i,j}} &= -\mathbf{Q} \left(\begin{smallmatrix} w_{j,f} \\ 0 \end{smallmatrix}\right) (\mathbf{f}^m_{i+L})^T \mathbf{Q}\, \mathbf{1}_{i \leq L} - \mathbf{Q} \left(\begin{smallmatrix} 0 \\ w_{j,p} \end{smallmatrix}\right) (\mathbf{f}^m_{i-L})^T \mathbf{Q}\, \mathbf{1}_{i > L} \\ &= -\mathbf{Q}\Pi_{pf} W \mathbf{e}_j \,(\mathbf{f}^m_i)^T \Pi_{pf}\mathbf{Q} - \mathbf{Q}\Pi_{fp} W \mathbf{e}_j \,(\mathbf{f}^m_i)^T \Pi_{fp}\mathbf{Q}. \end{aligned} \tag{3.16}$$

From this, we deduce immediately that

$$\frac{\partial \xi}{\partial \overline{W}^{m_1}_{i_1,j_1}} = -\frac{1}{ML} \Big(\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W + \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W \Big)^{m_1}_{i_1,j_1}.$$

Using that $\mathbb{E}\{W^{m_1}_{i_1,j_1} \overline{W}^{m_2}_{i_2,j_2}\} = \frac{1}{N} R_{m_1 m_2} \delta_{i_1+j_1, i_2+j_2}$, we obtain that ϕ is given by

$$\phi = \frac{1}{N(ML)^2} \sum_{\substack{i_1,j_1,m_1\\i_2,j_2,m_2}} (\mathbf{e}_{j_1})^T (\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W + \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)^* \mathbf{f}^{m_1}_{i_1} R_{m_1 m_2}$$
$$\times\, \delta_{i_1+j_1, i_2+j_2} (\mathbf{f}^{m_2}_{i_2})^T (\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W + \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W) \mathbf{e}_{j_2}.$$

We put $u = i_1 - i_2$ and remark that $\sum_{m_1,m_2,i_1-i_2=u} \mathbf{f}^{m_1}_{i_1} R_{m_1 m_2} (\mathbf{f}^{m_2}_{i_2})^T = J_L^{*\epsilon(u)} \otimes R$ and that $\sum_{j_2 - j_1 = u} \mathbf{e}_{j_2}\mathbf{e}^T_{j_1} = J_N^{*\epsilon(u)}$. Therefore, ϕ can be written as

$$\phi = \frac{1}{MLN} \mathbb{E}\Big\{ \sum_{u=-(L-1)}^{L-1} \frac{1}{ML} \mathrm{Tr} (\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W + \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)^* (J_L^{*\epsilon(u)} \otimes R)$$
$$\times\, (\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W + \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W) J_N^{*\epsilon(u)} \Big\}. \tag{3.17}$$

Each term inside the sum over u can be written as $\dfrac{1}{ML}\mathrm{Tr}A^*(I_L \otimes R^{1/2})(J_L^{*\epsilon(u)} \otimes I)(I_L \otimes R^{1/2}) A J_N^{*\epsilon(u)}$, where the expression of the $ML \times N$ matrix A is omitted. As $\|R\|$ is bounded by the nice constant b (see (2.2)), (3.8) and (3.17) lead to the conclusion that we just need to evaluate $\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}A^*A\}$. Using the Schwartz inequality, we obtain immediately that

$$\begin{aligned}\mathbb{E}\{\mathrm{Tr}A^*A\} \leq\, & 2\mathbb{E}\{\mathrm{Tr}\,((\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)^* \Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)\} \\ & + 2\mathbb{E}\{\mathrm{Tr}\,((\Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)^* \Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)\}. \end{aligned} \tag{3.18}$$

Since $(\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf})^* \,\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf} \leq \|\mathbf{Q}\|^4 \|F\|^2\, I$ and $\|\mathbf{Q}\| \leq \frac{1}{\mathrm{Im}z}$, we get that

$$\begin{aligned}\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\,((\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)^* \Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)\} &\leq \frac{1}{(\mathrm{Im}z)^4} \|F\|^2 \frac{1}{ML}\mathbb{E}\{\mathrm{Tr}W^*W\} \\ &\leq \frac{1}{(\mathrm{Im}z)^4} \|F\|^2\, \mathbb{E}\{\|W\|^2\} \end{aligned}$$

Lemma 3.1 thus implies that

$$\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\,((\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)^*\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)\} \leq \kappa^2 P\left(\frac{1}{\mathrm{Im}z}\right)$$

for some nice polynomial P. The term $\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\,(\Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)^*\Pi_{fp}\mathbf{Q}F\mathbf{Q}\Pi_{fp}W)\}$ can be handled similarly. Therefore, (3.17) leads to $\phi \leq \kappa^2\,\frac{1}{N^2}P\left(\frac{1}{\mathrm{Im}z}\right)$. This establishes (3.14).
To prove (3.15) one can also use Poincaré-Nash inequality for $\xi = \frac{1}{ML}\mathrm{Tr}F\mathbf{Q}GWHW^*$. After some calculations, we get that the variance of ξ is upperbounded by a term given by

$$\frac{\kappa_1}{N^2}\,\mathbb{E}\left(\frac{1}{ML}\mathrm{Tr}(F\mathbf{Q}GWH)^*(F\mathbf{Q}GWH) + \frac{1}{ML}\mathrm{Tr}(F\mathbf{Q}WH)^*(F\mathbf{Q}WH) + \eta_1 + \eta_2\right), \tag{3.19}$$

where κ_1 is some nice constant, and where η_1 and η_2 are defined by

$$\begin{aligned}
\eta_1 &= \frac{1}{ML}\mathrm{Tr}(\Pi_{pf}\mathbf{Q}GWHW^*F\mathbf{Q}\Pi_{pf}W)^*(\Pi_{pf}\mathbf{Q}GWHW^*F\mathbf{Q}\Pi_{pf}W),\\
\eta_2 &= \frac{1}{ML}\mathrm{Tr}(\Pi_{fp}\mathbf{Q}GWHW^*F\mathbf{Q}\Pi_{fp}W)^*(\Pi_{fp}\mathbf{Q}GWHW^*F\mathbf{Q}\Pi_{fp}W).
\end{aligned}$$

Using Lemma 3.1 as well as the inequality $\mathbf{Q}\mathbf{Q}^* \leq \frac{1}{\mathrm{Im}^2 z}I$, we obtain immediately (3.15). This completes the proof of Lemma 3.3. ■

In the following, we also need to evaluate the variance of more specific terms. The following result appears to be a consequence of Lemma 3.3 and of the particular structure (3.3) of matrix $\mathbf{Q}(z)$.

Corollary 3.1. *Let $(F_{1,N})_{N\geq 1}$ be a sequence of deterministic $ML \times ML$ matrices such that $\sup_N \|F_{1,N}\| \leq \kappa$, and $(H_N)_{N\geq 1}$ a sequence of deterministic $N \times N$ matrices satisfying $\sup_N \|H_N\| \leq 1$. Then, if $z \in \mathbb{C}^+$ and $\mathrm{Im}z^2 > 0$, the following evaluations hold :*

$$\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}F_1\mathbf{Q_{ij}}(z)\right\} \leq \kappa^2\frac{1}{N^2}P_1(|z^2|)P_2\left(\frac{1}{\mathrm{Im}z^2}\right), \tag{3.20}$$

where i and j belong to $\{p, f\}$;

$$\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}\left[HW^*\Pi_{i_1j_1}\begin{pmatrix} F_1 & 0\\ 0 & 0\end{pmatrix}\mathbf{Q}(z)\Pi_{i_2j_2}W\right]\right\} \leq \kappa^2\frac{1}{N^2}P_1(|z^2|)P_2\left(\frac{1}{\mathrm{Im}z^2}\right), \tag{3.21}$$

where i_1, j_1, i_2, j_2 still belong to $\{p, f\}$, but verify $i_1 \neq j_1$ and $i_2 \neq j_2$.

Proof. We first prove (3.20), and first consider the case where $i = j = p$. We define the $2ML \times 2ML$ matrix F by $F = \begin{pmatrix} F_1 & 0\\ 0 & 0\end{pmatrix}$, and remark that $\frac{1}{ML}\mathrm{Tr}F_1\mathbf{Q_{pp}}(z)$ coincides with $\xi = \frac{1}{ML}\mathrm{Tr}F\mathbf{Q}(z)$. We follow the proof of (3.14), and evaluate the right hand side of (3.18) in a more accurate manner by taking into account the particular structure of the present matrix F. It is easy to check that

$$\begin{aligned}
&\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\,(\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)^*\Pi_{pf}\mathbf{Q}F\mathbf{Q}\Pi_{pf}W)\}\\
&\qquad\qquad = \frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\left(W_f^*\mathbf{Q}_{\mathbf{pp}}^*F_1^*\mathbf{Q}_{\mathbf{fp}}^*\mathbf{Q}_{\mathbf{fp}}F_1\mathbf{Q}_{\mathbf{pp}}W_f\right)\}.
\end{aligned}$$

As $\mathbf{Q_{fp}}(z) = W_pW_f^*Q(z^2)$, we obtain that

$$\mathbf{Q}_{\mathbf{fp}}^*(z)\mathbf{Q}_{\mathbf{fp}}(z) = (Q(z^2))^*W_fW_p^*W_pW_f^*Q(z^2) \leq \|W\|^4\frac{1}{(\mathrm{Im}z^2)^2}\,I$$

if $\mathrm{Im}(z^2) > 0$. Therefore, it holds that

$$F_1^*\mathbf{Q}_{\mathbf{fp}}^*\mathbf{Q}_{\mathbf{fp}}F_1 \leq \kappa^2\|W\|^4\frac{1}{(\mathrm{Im}z^2)^2}\,I.$$

From this, using the expression of $\mathbf{Q}_{pp} = zQ(z^2)$, we obtain similarly that

$$W_f^* \mathbf{Q}_{\mathbf{pp}}^* F_1^* \mathbf{Q}_{\mathbf{fp}}^* \mathbf{Q}_{\mathbf{fp}} F_1 \mathbf{Q}_{\mathbf{pp}} W_f \leq \kappa^2 \|W\|^6 \frac{|z|^2}{(\mathrm{Im} z^2)^4}.$$

Lemma 3.1 thus leads to the conclusion that

$$\frac{1}{ML} \mathbb{E}\{\mathrm{Tr}\left(W_f^* \mathbf{Q}_{\mathbf{pp}}^* F_1^* \mathbf{Q}_{\mathbf{fp}}^* \mathbf{Q}_{\mathbf{fp}} F_1 \mathbf{Q}_{\mathbf{pp}} W_f\right)\} \leq \kappa^2 \frac{\kappa_1 |z|^2}{(\mathrm{Im} z^2)^4},$$

where κ_1 is a nice constant such that $\mathbb{E}(\|W_N\|^6) \leq \kappa_1$ for each N. Using similar arguments, we obtain that

$$\frac{1}{ML} \mathbb{E}\{\mathrm{Tr}\,(\Pi_{fp} \mathbf{Q} F \mathbf{Q} \Pi_{fp} W)^* \Pi_{fp} \mathbf{Q} F \mathbf{Q} \Pi_{fp} W)\} \leq \kappa^2 \frac{\kappa_1 |z^2|^2}{(\mathrm{Im} z^2)^4}.$$

This, in turn, implies (3.20) for $i = j = p$. As the arguments are essentially the same for the other values of i and j, we do not provide the corresponding proofs.

In order to establish (3.21), we follow the proof (3.15) for $F = \Pi_{i_1 j_1} \begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix}$, $G = \Pi_{i_2 j_2}$. It is necessary to check that the 4 terms inside the bracket of (3.19) can be upperbounded by $\kappa^2 P_1(|z^2|) P_2(\frac{1}{\mathrm{Im} z^2})$ for nice polynomials P_1 and P_2. As above, the use of the particular expression of matrices $(\mathbf{Q}_{\mathbf{ij}})_{i,j \in \{f,p\}}$ allows to establish this property. The corresponding easy calculations are omitted. ■

3.3 Various lemmas on Stieltjes transform

In this paragraph, we provide a number of useful results on certain Stieltjes transforms. We recall that if K is a positive integer, then a $K \times K$ matrix-valued positive measure ω is a σ–additive function from the Borel sets of $\mathbb{R}$ onto the set of all positive $K \times K$ matrices (see e.g. [42], Chapter 1 for more details). In the following, if A is a Borel set of $\mathbb{R}$, we denote by $\mathcal{S}_M(A)$ the set of all Stieltjes transforms of $M \times M$ matrix valued positive finite measures carried by A. $\mathcal{S}_1(A)$ is denoted $\mathcal{S}(A)$.
We now state a quite useful Lemma.

Lemma 3.4. *Let $\beta(z) \in \mathcal{S}(\mathbb{R}^+)$, and consider function $\boldsymbol{\beta}(z)$ defined by $\boldsymbol{\beta}(z) = z\beta(z^2)$. Then $\boldsymbol{\beta} \in \mathcal{S}(\mathbb{R})$. Moroever, it holds that*

$$\begin{aligned} \mathbf{G}(z) &= \left(-zI_M - \frac{c\boldsymbol{\beta}(z)}{1 - c^2\boldsymbol{\beta}^2(z)} R\right)^{-1} \in \mathcal{S}_M(\mathbb{R}) \\ G(z) &= \left(-zI_M - \frac{cz\beta(z)}{1 - zc^2\beta^2(z)} R\right)^{-1} \in \mathcal{S}_M(\mathbb{R}^+) \end{aligned} \tag{3.22}$$

and that

$$\mathbf{G}(z)\,(\mathbf{G}(z))^* \leq \frac{I_M}{(\mathrm{Im} z)^2},\ G(z)\,(G(z))^* \leq \frac{I_M}{(\mathrm{Im} z)^2}. \tag{3.23}$$

Finally, matrices $\mathbf{G}(z)$ and $G(z)$ are linked by the relation

$$\mathbf{G}(z) = zG(z^2) \tag{3.24}$$

for each $z \in \mathbb{C}^+$.

Proof. Let τ be the measure carried by $\mathbb{R}^+$ corresponding to the Stieltjes transform $\beta(z)$. We first prove that $\boldsymbol{\beta}(z)$ is a Stieltjes transform. We first remark that if $z \in \mathbb{C}^+$, then $z^2 \in \mathbb{C} - \mathbb{R}^+$. β analytic on $\mathbb{C} - \mathbb{R}^+$ thus implies that $\boldsymbol{\beta}(z)$ is analytic on $\mathbb{C}^+$. Moreover, it is clear that

$$\mathrm{Im}\boldsymbol{\beta}(z) = \mathrm{Im} \int_{\mathbb{R}^+} \frac{z d\tau(\lambda)}{\lambda - z^2} = \int_{\mathbb{R}^+} \frac{\mathrm{Im} z(\lambda + |z|^2) d\tau(\lambda)}{|\lambda - z^2|^2} > 0, \text{ when} \, \mathrm{Im} z > 0.$$

To evaluate $\boldsymbol{\beta}(z)$ for $z \in \mathbb{C}^+$, we write

$$\left|\int_{\mathbb{R}^+} \frac{z d\,\tau(\lambda)}{\lambda - z^2}\right| \leq \int_{\mathbb{R}^+} \frac{d\,\tau(\lambda)}{\left|\frac{\lambda}{z} - z\right|}.$$

Using that $\left|\frac{\lambda}{z} - z\right| \geq \left|\text{Im}(\frac{\lambda}{z} - z)\right| \geq \text{Im}z$ for $z \in \mathbb{C}^+$ and $\lambda \geq 0$, we get that

$$|\boldsymbol{\beta}(z)| \leq \int_{\mathbb{R}^+} \frac{d\,\tau(\lambda)}{\text{Im}z} = \frac{\tau(\mathbb{R}^+)}{\text{Im}z}.$$

From this and Proposition 2.1, we obtain that $\boldsymbol{\beta}(z) \in \mathcal{S}(\mathbb{R})$.
To prove (3.22), it is first necessary to show that $\mathbf{G}$ is analytic on $\mathbb{C}^+$. For this, we first check that $m(z) = 1 - c^2\boldsymbol{\beta}^2(z) \neq 0$ for $z \in \mathbb{C}^+$. Indeed, write $\boldsymbol{\beta}(z) = x + iy$ with $y > 0$, then $m(z) = 1 - c^2x^2 + c^2y^2 - 2cxyi$. Hence, if $x = 0$ we have $m(z) = 1 + c^2y^2 > 0$, and if $x \neq 0$ then $2xy \neq 0$ since $y > 0$. In order to establish that matrix $\left(-zI_M - \frac{c\boldsymbol{\beta}(z)}{1 - c^2\boldsymbol{\beta}^2(z)}R\right)$ is invertible on $\mathbb{C}^+$, we verify that

$$\text{Im}\left(-zI_M - \frac{c\boldsymbol{\beta}(z)}{1 - c^2\boldsymbol{\beta}^2(z)}R\right) < 0 \tag{3.25}$$

on $\mathbb{C}^+$. It is easy to check that

$$\text{Im}\left(-zI_M - \frac{c\boldsymbol{\beta}(z)}{1 - c^2\boldsymbol{\beta}^2(z)}R\right) = -\text{Im}z\, I_M - \frac{c\text{Im}\boldsymbol{\beta}(z)(1 + c^2|\boldsymbol{\beta}(z)|^2)}{|1 - c^2\boldsymbol{\beta}^2(z)|^2} R < -\text{Im}z\, I_M.$$

Therefore, $\text{Im}z > 0$ and $\text{Im}\boldsymbol{\beta}(z) > 0$ imply (3.25). The imaginary part of $\mathbf{G}(z)$ is given by

$$\text{Im}(\mathbf{G}(z)) = -\mathbf{G}(z)\text{Im}\left(-zI_M - \frac{c\boldsymbol{\beta}(z)}{1 - c^2\boldsymbol{\beta}^2(z)}R\right)(\mathbf{G}(z))^* > \text{Im}z\ (\mathbf{G}(z)\,(\mathbf{G}(z))^*) > 0.$$

Therefore, $\text{Im}\mathbf{G}(z) > 0$ if $z \in \mathbb{C}^+$. We finally remark that $\lim_{y \to +\infty} -iy\mathbf{G}(iy) = I_M$, which implies that $\sup_{y > \epsilon} \|iy\mathbf{G}(iy)\| < +\infty$ for each $\epsilon > 0$. Proposition 2.1 eventually implies that $\mathbf{G} \in \mathcal{S}_M(\mathbb{R})$. Moreover, if $\boldsymbol{\tau}^{\mathbf{G}}$ is the underlying $M \times M$ positive matrix valued measure, (2.5) leads to $\boldsymbol{\tau}^{\mathbf{G}}(\mathbb{R}) = I_M$.

We prove similarly the analyticity of $G(z)$ on $\mathbb{C}^+$. We first check that $1 - zc^2\beta^2(z) \neq 0$ if $z \in \mathbb{C}^+$, or equivalently that $|1 - zc^2\beta^2(z)| \neq 0$ if $z \in \mathbb{C}^+$. We remark that

$$|1 - zc^2\beta^2(z)| = |z\beta(z)||c^2\beta(z) - \frac{1}{z\beta(z)}| > \text{Im}z\,\text{Im}\beta(z)\,\text{Im}\left(c^2\beta(z) - \frac{1}{z\beta(z)}\right). \tag{3.26}$$

As $\beta \in \mathcal{S}(\mathbb{R}^+)$, it holds that $\text{Im}\left(c^2\beta(z) - \frac{1}{z\beta(z)}\right) > 0$ if $z \in \mathbb{C}^+$. Therefore, $1 - zc^2\beta^2(z) \neq 0$ if $z \in \mathbb{C}^+$. As above, we verify that

$$\text{Im}\left(-zI_M - \frac{cz\beta(z)}{1 - z(c\beta(z))^2}R\right) = -\text{Im}z\, I_M - \text{Im}\left(\frac{cz\beta(z)}{1 - z(c\beta(z))^2}\right) R < -\text{Im}z\, I_M. \tag{3.27}$$

For this, we remark that

$$\text{Im}\left(\frac{cz\beta(z)}{1 - z(c\beta(z))^2}\right) = \frac{c}{|1 - z(c\beta(z))^2|^2}\left(\text{Im}(z\beta(z)) + |zc\beta(z)|^2\text{Im}\beta(z)\right) > 0$$

if $z \in \mathbb{C}^+$, which, of course, leads to (3.27). Therefore, matrix $\left(-zI_M - \frac{cz\beta(z)}{1 - z(c\beta(z))^2}R\right)$ is invertible if $z \in \mathbb{C}^+$, and G is analytic on $\mathbb{C}^+$. Moreover, we obtain immediately that

$$\text{Im}(G(z)) = G(z)\left(\text{Im}z\, I_M + \text{Im}\left(\frac{cz\beta(z)}{1 - z(c\beta(z))^2}\right) R\right)(G(z))^* > \text{Im}z\ (G(z)G(z)^*) > 0 \tag{3.28}$$

$$\text{Im}(zG(z)) = G(z)\text{Im}\left(\frac{cz\beta(z)}{1 - z(c\beta(z))^2}\right) R(G(z))^* > 0$$

for $z \in \mathbb{C}^+$. As above, it holds that $\lim_{y \to +\infty} -iyG(iy) = I$ and that $\sup_{y > \epsilon} \|iyG(iy)\| < +\infty$ for each $\epsilon > 0$. This implies that $G \in \mathcal{S}_M(\mathbb{R}^+)$, and that if τ^G represents the associated $M \times M$ matrix-valued measure, then $\tau^G(\mathbb{R}^+) = I$.
In order to establish (3.23), we follow [17, Lemma 3.1]. More precisely, we remark that

$$\mathrm{Im}G(z) = \mathrm{Im}z \int_{\mathbb{R}^+} \frac{d\tau^G(\lambda)}{|\lambda - z|^2} < \frac{\tau^G(\mathbb{R}^+)}{\mathrm{Im}z} = \frac{I}{\mathrm{Im}z}.$$

Therefore, (3.28) leads to $(G(z)G(z)^*) \leq \frac{I}{(\mathrm{Im}z)^2}$. The other statement of (3.23) is proved similarly and this completes the proof. ■

Lemma 3.5. *We consider a sequence $(\beta_N)_{N \geq 1}$ of elements of $\mathcal{S}(\mathbb{R}^+)$ whose associated positive measures $(\tau_N)_{N \geq 1}$ satisfy for each $N \geq 1$*

$$\tau_N(\mathbb{R}^+) = \frac{1}{M} \mathrm{Tr} R_N \tag{3.29}$$

as well as

$$\int_{\mathbb{R}^+} \lambda \, d\tau_N(\lambda) = c_N \frac{1}{M} \mathrm{Tr} R_N \frac{1}{M} \mathrm{Tr} R_N^2. \tag{3.30}$$

Then, it exist nice constants ω, κ such that

$$\mathrm{Im}\beta_N(z) \geq \frac{\kappa \, \mathrm{Im}z}{(\omega^2 + |z|^2)} \tag{3.31}$$

and

$$\left|1 - z\,(c_N \beta_N(z))^2\right| \geq \frac{\kappa\,(\mathrm{Im}z)^3}{(\omega^2 + |z|^2)^2} \tag{3.32}$$

for each $z \in \mathbb{C}^+$ and for each $N \geq 1$. Moreover, if $\boldsymbol{\beta}_N(z)$ is defined by $\boldsymbol{\beta}_N(z) = z\,\beta_N(z^2)$, then, we also have

$$\mathrm{Im}\boldsymbol{\beta}_N(z) \geq \frac{\kappa\;(\mathrm{Im}z)^3}{(\omega^2 + |z|^4)} \tag{3.33}$$

and

$$\left|1 - (c_N \boldsymbol{\beta}_N(z))^2\right| \geq \frac{\kappa\,(\mathrm{Im}z)^6}{(\omega^2 + |z|^4)^2} \tag{3.34}$$

for each $z \in \mathbb{C}^+$ and for each $N \geq 1$.

Proof. We first establish (3.31). $\mathrm{Im}\beta_N(z)$ is given by

$$\mathrm{Im}\beta_N(z) = \mathrm{Im}z \int_{\mathbb{R}^+} \frac{d\,\tau_N(\lambda)}{|\lambda - z|^2}.$$

For each $\omega > 0$, it is clear that

$$\int_{\mathbb{R}^+} \frac{d\,\tau_N(\lambda)}{|\lambda - z|^2} \geq \int_0^\omega \frac{d\,\tau_N(\lambda)}{|\lambda - z|^2} \geq \frac{\tau_N([0,\omega])}{2(\lambda^2 + |z|^2)}.$$

Assumption (2.2) and (3.30) imply that the sequence $(\tau_N)_{N \geq 1}$ is tight. For each $\epsilon > 0$, it thus exists $\omega > 0$ for which $\tau_N(]\omega, +\infty[) < \epsilon$ for each N, or equivalently, $\tau_N([0,\omega]) > \tau_N(\mathbb{R}^+) - \epsilon$. As $\tau_N(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}(R_N) > a$, we choose $\epsilon = a/2$, and obtain that the corresponding ω verifies $\tau_N([0,\omega]) > a/2$ for each N. This completes the proof of (3.31). We now verify (3.32). For this, we use (3.26). As $\mathrm{Im}\left(\frac{1}{z\beta_N(z)}\right) < 0$, it holds that $\mathrm{Im}\left(c_N^2 \beta_N(z) - \frac{1}{z\beta_N(z)}\right) \geq c_N^2 \mathrm{Im}\beta_N(z)$. Therefore, we obtain that

$$\left|1 - z\,(c_N \beta_N(z))^2\right| \geq c_N^2 \,\mathrm{Im}z\; (\mathrm{Im}\beta_N(z))^2 \tag{3.35}$$

which implies (3.32).

We finally verify (3.33) and (3.34). For this, we first express $\boldsymbol{\beta}_N(z)$ as

$$\boldsymbol{\beta}_N(z) = z\beta_N(z^2) = \int_{\mathbb{R}^+} \frac{z}{\lambda - z^2} d\,\tau_N(\lambda)$$

which leads immediately to

$$\begin{aligned}\mathrm{Im}\boldsymbol{\beta}_N(z) = \mathrm{Im}z \int_{\mathbb{R}^+} \frac{\lambda + |z|^2}{|\lambda - z^2|^2} d\,\tau_N(\lambda) &\geq \mathrm{Im}z\, |z|^2 \int_{\mathbb{R}^+} \frac{1}{|\lambda - z^2|^2} d\,\tau_N(\lambda) \\ &\geq (\mathrm{Im}z)^3 \int_{\mathbb{R}^+} \frac{1}{|\lambda - z^2|^2} d\,\tau_N(\lambda).\end{aligned}$$

We observe that for $\omega > 0$, then,

$$\int_{\mathbb{R}^+} \frac{1}{|\lambda - z^2|^2} d\,\tau_N(\lambda) \geq \int_0^\omega \frac{1}{|\lambda - z^2|^2} d\,\tau_N(\lambda) \geq \frac{1}{2(\omega^2 + |z|^4)}\, \tau_N([0, \omega]).$$

As justified above, it is possible to choose ω for which $\tau_N([0, \omega]) \geq \frac{a}{2}$ for each N. This leads to (3.33). We now remark that $|1 - c_N^2 \boldsymbol{\beta}_N^2| = |\boldsymbol{\beta}_N| |\frac{1}{\boldsymbol{\beta}_N} - c_N^2 \boldsymbol{\beta}_N|$. As $\mathrm{Im}\boldsymbol{\beta}_N > 0$ on $\mathbb{C}^+$, it holds that

$$\left| \frac{1}{\boldsymbol{\beta}_N} - c_N^2 \boldsymbol{\beta}_N \right| \geq \left| \mathrm{Im} \left(\frac{1}{\boldsymbol{\beta}_N} - c_N^2 \boldsymbol{\beta}_N \right) \right| \geq c_N^2 \mathrm{Im}\boldsymbol{\beta}_N.$$

Using that $|\boldsymbol{\beta}_N| \geq \mathrm{Im}\boldsymbol{\beta}_N$, we eventually obtain that

$$|1 - c_N^2 \boldsymbol{\beta}_N^2| \geq c_N^2 \ (\mathrm{Im}\boldsymbol{\beta}_N)^2$$

which, in turn, implies (3.34). ■

3.4 Expression of matrix $\mathbb{E}\{\mathbf{Q}\}$ obtained using the integration by parts formula

We now express $\mathbb{E}\{\mathbf{Q}(z)\}$ using the integration by parts formula and deduce from this an approximate expression of $\mathbb{E}(Q(z))$. For this, we have first to establish some useful properties of $\mathbb{E}\{\mathbf{Q}(z)\}$ that follow from the invariance properties of the probability distribution of the observations $(y_n)_{n=1,\ldots,N}$. In the following, for $k, l \in \{1, 2, \ldots, L\}$, we denote by $\mathbf{Q}_{\mathbf{pp}}^{k,l}$ and $\mathbf{Q}_{\mathbf{ff}}^{k,l}$ the $M \times M$ matrices whose entries are given by $\left(\mathbf{Q}_{\mathbf{pp}}^{k,l}\right)_{m,n} = (\mathbf{Q}_{\mathbf{pp}})_{(k-1)M+m,(l-1)M+n}$ and $\left(\mathbf{Q}_{\mathbf{ff}}^{k,l}\right)_{m,n} = (\mathbf{Q}_{\mathbf{ff}})_{(k-1)M+m,(l-1)M+n}$for each $m, n \in \{1, 2, \ldots, M\}$.

Lemma 3.6. *The matrices $\mathbb{E}\{\mathbf{Q}_{\mathbf{pp}}\}$ and $\mathbb{E}\{\mathbf{Q}_{\mathbf{ff}}\}$ are block diagonal, i.e. $\mathbb{E}\left(\mathbf{Q}_{\mathbf{pp}}^{k,l}\right) = \mathbb{E}\{\mathbf{Q}_{\mathbf{ff}}^{k,l}\} = 0$ if $k \neq l$, and*

$$\mathrm{Tr}\mathbb{E}\{\mathbf{Q}_{\mathbf{pp}}\}(I_L \otimes R) = \mathrm{Tr}\mathbb{E}\{\mathbf{Q}_{\mathbf{ff}}\}(I_L \otimes R), \tag{3.36}$$

$$\mathbb{E}\{\mathbf{Q}_{\mathbf{pf}}\} = \mathbb{E}\{\mathbf{Q}_{\mathbf{fp}}\} = 0. \tag{3.37}$$

Proof. To prove (3.37), we consider the new set of vectors $z_k = e^{-ik\theta} y_k$ and construct the matrices Z_p, Z_f in the same way as Y_p and Y_f. It is clear that sequence $(z_n)_{n \in \mathbb{Z}}$ has the same probability distribution that $(y_n)_{n \in \mathbb{Z}}$. Z_p and Z_f can be expressed as

$$Z_p = \begin{pmatrix} e^{-i\theta} I_M & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{-Li\theta} I_M \end{pmatrix} Y_p \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{-(N-1)i\theta} \end{pmatrix},$$

$$Z_f = e^{-Li\theta} \begin{pmatrix} e^{-i\theta} I_M & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{-Li\theta} I_M \end{pmatrix} Y_f \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{-(N-1)i\theta} \end{pmatrix}.$$

Therefore, it holds that

$$Z_f Z_p^* Z_p Z_f^* = \begin{pmatrix} e^{-i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta} I_M \end{pmatrix} Y_f Y_p^* Y_p Y_f^* \begin{pmatrix} e^{i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{Li\theta} I_M \end{pmatrix}.$$

Similarly to $\mathbf{Q}$ we define matrix $\mathbf{Q^Z} = \begin{pmatrix} -zI_{ML} & \frac{1}{N} Z_f Z_p^* \\ \frac{1}{N} Z_p Z_f^* & -zI_{ML} \end{pmatrix}^{-1}$ and obtain immediately that

$$\mathbb{E}\{\mathbf{Q^Z_{pp}}\} = \begin{pmatrix} e^{-i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta} I_M \end{pmatrix} \mathbb{E}\{\mathbf{Q_{pp}}\} \begin{pmatrix} e^{i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{Li\theta} I_M \end{pmatrix}.$$

Since $\mathbb{E}\{\mathbf{Q^Z_{pp}}\} = \mathbb{E}\{\mathbf{Q_{pp}}\}$, then for any $M \times M$ block $\mathbb{E}\{\mathbf{Q_{pp}}^{j,k}\}$, we have

$$\mathbb{E}\{\mathbf{Q_{pp}}^{j,k}\} = e^{-ji\theta} \mathbb{E}\{\mathbf{Q_{pp}}^{j,k}\} e^{ki\theta} = e^{(k-j)i\theta} \mathbb{E}\{\mathbf{Q_{pp}}^{j,k}\}.$$

This proves that $\mathbb{E}\{\mathbf{Q_{pp}}^{j,k}\} = 0$ if $k \neq j$ as expected. A similar proof leads to the conclusion that $\mathbb{E}\{\mathbf{Q_{ff}}\}$ is block diagonal. Moroever, the equality $\mathbb{E}\{\mathbf{Q^Z_{fp}}\} = \mathbb{E}\{\mathbf{Q_{fp}}\}$ implies that

$$\mathbb{E}\{\mathbf{Q^Z_{fp}}\} = e^{-Li\theta} \begin{pmatrix} e^{-i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta} I_M \end{pmatrix} \mathbb{E}\{\mathbf{Q_{fp}}\} \begin{pmatrix} e^{i\theta} I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{Li\theta} I_M \end{pmatrix}.$$

Therefore, each $M \times M$ block $\mathbf{Q_{fp}}^{j,k}$ of $\mathbf{Q_{fp}}$ verifies $\mathbb{E}\{\mathbf{Q_{fp}}^{j,k}\} = e^{-(L+j-k)i\theta} \mathbb{E}\{\mathbf{Q_{fp}}^{j,k}\}$. As $j - k \in \{-(L-1), \dots, L-1\}$, this implies that $\mathbb{E}\{\mathbf{Q_{fp}}^{j,k}\} = 0$. This leads immediately to $\mathbb{E}\{\mathbf{Q_{fp}}\} = 0$. We obtain similarly that $\mathbb{E}\{\mathbf{Q_{pf}}\} = 0$.
To prove (3.36) we consider the sequence z defined by $z_n = y_{-n+N+2L}$ for each n. Again, the distribution of z_n will remain the same and it is easy to see that Z_p and Z_f are given by

$$Z_f = \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix} Y_p \begin{pmatrix} 0 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix},$$
$$Z_p = \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix} Y_f \begin{pmatrix} 0 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix}.$$

From this, we obtain that

$$\mathbb{E}\{\mathbf{Q^Z_{pp}}\} = \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix} \mathbb{E}\{\mathbf{Q_{ff}}\} \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix}.$$

As $\mathbb{E}\{\mathbf{Q^Z_{pp}}\} = \mathbb{E}\{\mathbf{Q_{pp}}\}$, this immediately implies that $\mathbb{E}\{\mathbf{Q_{ff}}^{j,j}\} = \mathbb{E}\{\mathbf{Q_{pp}}^{L-j,L-j}\}$, and, as a consequence, that $\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pp}}(I_L \otimes R)\} = \mathbb{E}\{\mathrm{Tr}\mathbf{Q_{ff}}(I_L \otimes R)\}$, as expected. ■

In order to present the following approximation of $\mathbb{E}(Q_N(z))$, we introduce some useful notations. $\alpha_N(z)$ is the function defined by

$$\alpha_N(z) = \frac{1}{ML} \mathrm{Tr}\left(\mathbb{E}\{Q_N(z)(I_L \times R_N)\}\right). \tag{3.38}$$

α_N is clearly an element of $\mathcal{S}(\mathbb{R}^+)$. In order to evaluate its associated positive measure $\overline{\mu}_N$, we denote by $\hat{\mu}_N$ the positive measure defined by

$$d\hat{\mu}_N(\lambda) = \frac{1}{ML} \sum_{i=1}^{ML} \hat{f}_i^* (I_L \otimes R) \hat{f}_i \, \delta_{\hat{\lambda}_i}, \tag{3.39}$$

where we recall that $(\hat{\lambda}_i)_{i=1,\dots,ML}$ and $(\hat{f}_i)_{i=1,\dots,ML}$ represent the eigenvalues and eigenvectors of $W_f W_p^* W_p W_f^*$. We remark that $\hat{\mu}_N$ is carried by $\mathbb{R}^+$ and that its mass $\hat{\mu}_N(\mathbb{R}^+)$ coincides with $\frac{1}{M}\mathrm{Tr}R_N$. Then, measure $\overline{\mu}_N$ is defined by

$$\int_{\mathbb{R}^+} \phi(\lambda)\, d\overline{\mu}_N(\lambda) = \mathbb{E}\left(\int_{\mathbb{R}^+} \phi(\lambda)\, d\hat{\mu}_N(\lambda)\right) \tag{3.40}$$

and satisfies $\overline{\mu}_N(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}R_N$. We also define $\boldsymbol{\alpha}_N(z)$ as the function

$$\boldsymbol{\alpha}_N(z) = z\alpha_N(z^2) \tag{3.41}$$

which, due to the identity $\mathbf{Q}_{\mathbf{pp}}(z) = zQ(z^2)$, is also given by

$$\boldsymbol{\alpha}_N(z) = \frac{1}{ML}\mathbb{E}\left\{\mathrm{Tr}\mathbf{Q}_{\mathbf{N},\mathbf{pp}}(z)(I_L \otimes R_N)\right\}. \tag{3.42}$$

Lemma 3.4 implies that $\boldsymbol{\alpha}_N \in \mathcal{S}(\mathbb{R})$ and that the $M \times M$ matrix-valued functions $S_N(z)$ and $\mathbf{S}_N(z)$ defined by

$$S_N(z) = -\left(zI_M + \frac{c_N z\alpha_N(z)}{1 - c_N^2 z\alpha_N(z)^2} R_N\right)^{-1} \tag{3.43}$$

and

$$\mathbf{S}_N(z) = -\left(\frac{c_N \boldsymbol{\alpha}(z)}{1 - c_N^2 \boldsymbol{\alpha}^2(z)} R + z\right)^{-1} = zS_N(z^2) \tag{3.44}$$

belong to $\mathcal{S}_M(\mathbb{R}^+)$ and $\mathcal{S}_M(\mathbb{R})$ respectively. We are now in position to introduce the main result of this section.

Theorem 3.1. *The matrix $\mathbb{E}(Q_N(z))$ can be written as*

$$\mathbb{E}\{Q_N(z)\} = I_L \otimes S_N(z) - E_N(z)\ (I_L \otimes S_N(z)), \tag{3.45}$$

where $E_N(z)$ is an error term such that

$$\left|\frac{1}{ML}\mathrm{Tr}E_N(z)F_N\right| \le \kappa\, \frac{1}{N^2}\, P_1(|z|)P_2(\frac{1}{\mathrm{Im}z}) \tag{3.46}$$

for each $z \in \mathbb{C}^+$ and for each deterministic $ML \times ML$ sequence of matrices $(F_N)_{N\ge 1}$ such that $\sup_{N\ge 1}\|F_N\| \le \kappa$.

In order to establish Theorem 3.1, we express $\mathbb{E}\{\mathbf{Q}(z)\}$ for $z \in \mathbb{C}^+$ by using the integration by parts formula (see Proposition 2.2), and deduce from that the expression (3.45) of $\mathbb{E}\{Q(z)\}$. The properties of the error term $E_N(z)$ is finally deduced from the results of section 3.2.

We recall that matrix $\mathbf{M}$ is defined by (3.2). In order to express $\mathbb{E}\{\mathbf{Q}(z)\}$ for $z \in \mathbb{C}^+$, we use the identity

$$z\mathbf{Q}(z) = -I_{2ML} + \mathbf{Q}(z)\mathbf{M} = -I_{2ML} + \sum_{j=1}^N \mathbf{Q}(z)\begin{pmatrix} 0 & w_{f,j}w_{p,j}^* \\ w_{p,j}w_{f,j}^* & 0 \end{pmatrix}. \tag{3.47}$$

For every $m_1, m_2 = 1, \dots, M$, $i_1 = 1, \dots, 2L$ and $i_2 = 1, \dots, L$ we denote by $\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}$ the $2N \times 2N$ matrix defined by

$$\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2} = \begin{pmatrix} \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pp) & \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \\ \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(fp) & \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff) \end{pmatrix}, \tag{3.48}$$

where the 4 $N \times N$ blocks are given by

$$\begin{aligned}
(\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf))_{jk} &= \left(\mathbf{Q}\left(\begin{smallmatrix} 0 \\ w_{p,j}\end{smallmatrix}\right)\right)_{i_1}^{m_1} (w_{f,k}^*)_{i_2}^{m_2}, \\
(\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pp))_{jk} &= \left(\mathbf{Q}\left(\begin{smallmatrix} 0 \\ w_{p,j}\end{smallmatrix}\right)\right)_{i_1}^{m_1} (w_{p,k}^*)_{i_2}^{m_2}, \\
(\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff))_{jk} &= \left(\mathbf{Q}\left(\begin{smallmatrix} w_{f,j} \\ 0\end{smallmatrix}\right)\right)_{i_1}^{m_1} (w_{f,k}^*)_{i_2}^{m_2}, \\
(\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(fp))_{jk} &= \left(\mathbf{Q}\left(\begin{smallmatrix} w_{f,j} \\ 0\end{smallmatrix}\right)\right)_{i_1}^{m_1} (w_{p,k}^*)_{i_2}^{m_2}.
\end{aligned} \tag{3.49}$$

We also define matrix $\mathbf{A}_{i_1i_2}^{m_1m_2}$ by $\mathbf{A}_{i_1i_2}^{m_1m_2} = \mathbb{E}\{\hat{\mathbf{A}}_{i_1i_2}^{m_1m_2}\}$. (3.47) implies that

$$z\mathbb{E}\{\mathbf{Q}_{i_1i_2}^{m_1m_2}(z)\} = -\delta_{i_1,i_2}\delta_{m_1,m_2} + \mathrm{Tr}\mathbf{A}_{i_1i_2}^{m_1m_2}(pf)\mathbf{1}_{i_2\le L} + \mathrm{Tr}\mathbf{A}_{i_1i_2-L}^{m_1m_2}(fp)\mathbf{1}_{i_2>L}. \tag{3.50}$$

In the reminder of this paragraph, we evaluate for each i_1, i_2, m_1, m_2 the elements of matrix $\mathbf{A}_{i_1i_2}^{m_1m_2}$ using (2.6) and (3.16). As we shall see, each element of $\mathbf{A}_{i_1i_2}^{m_1m_2}$ can be written as a functional of matrix $\mathbb{E}\{\mathbf{Q}\}$ plus an error term whose contribution vanishes when $N \to +\infty$. Plugging these expressions of $\mathbf{A}_{i_1i_2}^{m_1m_2}$ into (3.50) will establish an approximate expression of $\mathbb{E}\{\mathbf{Q}\}$. As the calculations are very tedious, we just indicate how each element $(\mathbf{A}_{i_1i_2}^{m_1m_2}(ff))_{j,k}$ of matrix $\mathbf{A}_{i_1i_2}^{m_1m_2}(ff)$ can be evaluated. By using integration by parts formula (2.6) and (3.16) we obtain

$$\mathbb{E}\left\{\left(\mathbf{Q}\begin{pmatrix} w_{f,j} \\ 0\end{pmatrix}\right)_{i_1}^{m_1} (w_{f,k}^*)_{i_2}^{m_2}\right\} = \sum_{i_3=1}^{L}\sum_{m_3}\mathbb{E}\{\mathbf{Q}_{i_1i_3}^{m_1m_3}W_{i_3+L,j}^{m_3}\overline{W}_{i_2+L,k}^{m_2}\}$$

$$= \sum_{i_3=1}^{L}\sum_{\substack{i',j'\\ m',m_3}}\mathbb{E}\{W_{i_3+L,j}^{m_3}\overline{W}_{i',j'}^{m'}\} \times \mathbb{E}\left\{\frac{\partial\left(\mathbf{Q}_{i_1i_3}^{m_1m_3}\overline{W}_{i_2+L,k}^{m_2}\right)}{\partial \overline{W}_{i',j'}^{m'}}\right\} = \frac{1}{N}\sum_{i_3=1}^{L}\sum_{\substack{i',j'\\ m',m_3}} R_{m_3m'}$$

$$\times\, \delta_{i_3+L+j,i'+j'}\mathbb{E}\left\{\mathbf{Q}_{i_1i_3}^{m_1m_3}\delta_{m_2,m'}\delta_{i_2+L,i'}\delta_{k,j'} + \overline{W}_{i_2+L,k}^{m_2}\frac{\partial \mathbf{Q}_{i_1i_3}^{m_1m_3}}{\partial \overline{W}_{i',j'}^{m'}}\right\}$$

$$= \frac{1}{N}\sum_{i_3=1}^{L}\sum_{m_3=1}^{M}\mathbb{E}\left\{\mathbf{Q}_{i_1i_3}^{m_1m_3}R_{m_3m_2}\delta_{i_3,i_2-(j-k)}\right\} - \frac{1}{N}\sum_{\substack{i_3,j'\\ m_3,m'}}\sum_{i'=1}^{L}R_{m_3m'}\delta_{i_3+L+j,i'+j'}$$

$$\times\,\mathbb{E}\left\{\overline{W}_{i_2,k}^{(f)m_2}\left(\mathbf{Q}\begin{pmatrix} w_{f,j'} \\ 0\end{pmatrix}\right)_{i_1}^{m_1}\mathbf{Q}_{i'+Li_3}^{m'm_3}\right\} - \frac{1}{N}\sum_{\substack{i_3,j'\\ m_3,m'}}\sum_{i'=L+1}^{2L}R_{m_3m'}\delta_{i_3+L+j,i'+j'}$$

$$\times\,\mathbb{E}\left\{\overline{W}_{i_2,k}^{(f)m_2}\left(\mathbf{Q}\begin{pmatrix} 0 \\ w_{p,j'}\end{pmatrix}\right)_{i_1}^{m_1}\mathbf{Q}_{i'-Li_3}^{m'm_3}\right\} = \frac{1}{N}\sum_{i_3=1}^{L}\mathbb{E}\Big\{\left(\begin{pmatrix}\mathbf{Q_{pp}}\\ \mathbf{Q_{fp}}\end{pmatrix}(I_L\otimes R)\right)_{i_1i_3}^{m_1m_2}$$

$$\times\,\delta_{i_3,i_2-(j-k)}\Big\} - \frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L}\delta_{i_3+L+j,i'+j'}\mathbb{E}\left\{\left(\hat{A}_{i_1i_2}^{m_1m_2}(ff)\right)_{j',k}(\mathbf{Q_{fp}}(I_L\otimes R))_{i'i_3}^{m'm'}\right\}$$

$$-\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L}\delta_{i_3+j,i'+j'}\mathbb{E}\left\{\left(\hat{\mathbf{A}}_{i_1i_2}^{m_1m_2}(pf)\right)_{j',k}(\mathbf{Q_{pp}}(I_L\otimes R))_{i'i_3}^{m'm'}\right\}.$$

Now we define for every $i_1 = 1,\dots,2L$, $i_2 = 1,\dots,L$ and $m_1, m_2 = 1,\dots,M$ $2N\times 2N$ matrix $\mathbf{B}_{i_1i_2}^{m_1m_2}$ with $N\times N$ blocks

$$\left(\mathbf{B}_{i_1i_2}^{m_1m_2}(fp)\right)_{j,k} = \frac{1}{N}\mathbb{E}\left\{\begin{pmatrix}\mathbf{Q_{pp}}\\ \mathbf{Q_{fp}}\end{pmatrix}(I_L\otimes R)\right\}_{i_1,i_2-(j-k)-L}^{m_1,m_2}\mathbf{1}_{1\le i_2-(j-k)-L\le L},$$

$$\left(\mathbf{B}_{i_1i_2}^{m_1m_2}(ff)\right)_{j,k} = \frac{1}{N}\mathbb{E}\left\{\begin{pmatrix}\mathbf{Q_{pp}}\\ \mathbf{Q_{fp}}\end{pmatrix}(I_L\otimes R)\right\}_{i_1,i_2-(j-k)}^{m_1,m_2}\mathbf{1}_{1\le i_2-(j-k)\le L},$$

$$\left(\mathbf{B}_{i_1i_2}^{m_1m_2}(pp)\right)_{j,k} = \frac{1}{N}\mathbb{E}\left\{\begin{pmatrix}\mathbf{Q_{pf}}\\ \mathbf{Q_{ff}}\end{pmatrix}(I_L\otimes R)\right\}_{i_1,i_2-(j-k)}^{m_1,m_2}\mathbf{1}_{1\le i_2-(j-k)\le L},$$

$$\left(\mathbf{B}_{i_1i_2}^{m_1m_2}(pf)\right)_{j,k} = \frac{1}{N}\mathbb{E}\left\{\begin{pmatrix}\mathbf{Q_{pf}}\\ \mathbf{Q_{ff}}\end{pmatrix}(I_L\otimes R)\right\}_{i_1,i_2-(j-k)+L}^{m_1,m_2}\mathbf{1}_{1\le i_2-(j-k)+L\le L}.$$

For every $ML\times ML$ block matrix $\mathbf{D}$, we define the sequence $(\tau^{(M)}(\mathbf{D})(l))_{l=-L+1,\dots,L-1}$ as

$$\tau^{(M)}(\mathbf{D})(l) = \frac{1}{ML}\mathrm{Tr}\mathbf{D}(J_L^{(l)}\otimes I_M) = \frac{1}{ML}\sum_{m=1}^{M}\sum_{i-i'=l}\mathbf{D}_{i,i'}^{m,m} \tag{3.51}$$

and the $N \times N$ Toeplitz matrix $\mathcal{T}_{N,L}^{(M)}(\mathbf{D})$ given by

$$\mathcal{T}_{N,L}^{(M)}(\mathbf{D}) = \sum_{l=-L+1}^{L-1} \tau^{(M)}(\mathbf{D})(l) J_N^{*\epsilon(l)}. \tag{3.52}$$

In other words, the entries of $\mathcal{T}_{N,L}^{(M)}(\mathbf{D})$ are defined by the relation

$$\left[\mathcal{T}_{N,L}^{(M)}(\mathbf{D})\right]_{j_1,j_2} = \tau^{(M)}(\mathbf{D})(j_1 - j_2)\, \mathbf{1}_{-(L-1)\leq j_1-j_2\leq L-1}. \tag{3.53}$$

We observe that if $\mathbf{D}$ is block diagonal, i.e. if $\mathbf{D}_{i_1,i_2}^{m_1,m_2} = 0$ for each m_1, m_2 when $i_1 \neq i_2$, then, matrix $\mathcal{T}_{N,L}^{(M)}(\mathbf{D})$ coincides with the diagonal matrix $\mathcal{T}_{N,L}^{(M)}(\mathbf{D}) = \left(\frac{1}{ML}\mathrm{Tr}\mathbf{D}\right) I_N$. It clear that

$$\frac{1}{N}\sum_{i_3=1}^{L} \mathbb{E}\left\{ \left(\begin{pmatrix} \mathbf{Q_{pp}} \\ \mathbf{Q_{fp}} \end{pmatrix} (I_L \otimes R) \right)_{i_1 i_3}^{m_1 m_2} \delta_{i_3, i_2-(j-k)} \right\} = \left(\mathbf{B}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j,k}.$$

In order to rewrite the term

$$\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L} \delta_{i_3+L+j,i'+j'} \times \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j',k} (\mathbf{Q_{fp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\}$$

in a more convenient way, we put $l = i' - i_3$, and remark that

$$\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L} \delta_{i_3+L+j,i'+j'} \times \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j',k} (\mathbf{Q_{fp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\} =$$
$$\frac{ML}{N}\sum_{m'}\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{L+j-l,k} \frac{1}{ML}\sum_{i'-i_3=l} (\mathbf{Q_{fp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\}.$$

Using the definition (3.51), this can be rewritten as

$$\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L} \delta_{i_3+L+j,i'+j'} \times \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j',k} (\mathbf{Q_{fp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\} =$$
$$c_N \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{L+j-l,k} \tau^M \left(\mathbf{Q_{fp}}(I_L \otimes R) \right)(l) \right\}.$$

We introduce $j' = L + j - l$, and using (3.53), we notice that

$$\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L} \delta_{i_3+L+j,i'+j'} \times \mathbb{E}\left\{ \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j',k} (\mathbf{Q_{fp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\} =$$
$$c_N\, \mathbb{E}\left\{ \sum_{j'=1}^{N} \left[\mathcal{T}_{N,L}^{(M)} \left(\mathbf{Q_{fp}}(I_L \otimes R) \right) \right]_{L+j,j'} \left(\hat{A}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j',k} \right\} =$$
$$c_N \mathbb{E}\left\{ \left(J_N^L \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{fp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j,k} \right\}.$$

We obtain similarly that

$$\frac{1}{N}\sum_{m',j'}\sum_{i_3,i'=1}^{L} \delta_{i_3+j,i'+j'} \mathbb{E}\left\{ \left(\hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \right)_{j',k} (\mathbf{Q_{pp}}(I_L \otimes R))_{i' i_3}^{m'm'} \right\} =$$
$$c_N \mathbb{E}\left\{ \left(\mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \right)_{j,k} \right\}.$$

Therefore, matrix $\mathbf{A}_{i_1 i_2}^{m_1 m_2}(ff)$ is also given by

$$\left(\mathbf{A}_{i_1 i_2}^{m_1 m_2}(ff)\right)_{j,k} = \left(\mathbf{B}_{i_1 i_2}^{m_1 m_2}(ff)\right)_{j,k} - c_N \mathbb{E}\left\{ \left(J_N^L \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{fp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j,k} \right\}$$
$$- c_N \mathbb{E}\left\{ \left(\mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \right)_{j,k} \right\}.$$

Writing $\mathbf{Q_{fp}}$ and $\mathbf{Q_{pp}}$ as $\mathbf{Q_{fp}} = \mathbb{E}\{\mathbf{Q_{fp}}\} + \mathbf{Q}^\circ_{\mathbf{fp}} = \mathbf{Q}^\circ_{\mathbf{fp}}$ (see (3.37)) and $\mathbf{Q_{pp}} = \mathbb{E}\{\mathbf{Q_{pp}}\} + \mathbf{Q}^\circ_{\mathbf{pp}}$, we obtain that

$$\left(\mathbf{A}_{i_1 i_2}^{m_1 m_2}(ff)\right)_{j,k} = \left(\mathbf{B}_{i_1 i_2}^{m_1 m_2}(ff)\right)_{j,k} - c_N \mathbb{E}\left\{ \left(\mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R)) \mathbf{A}_{i_1 i_2}^{m_1 m_2}(pf) \right)_{j,k} \right\}$$
$$- c_N \mathbb{E}\left\{ \left(J_N^L \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{fp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff) \right)_{j,k} \right\}$$
$$- c_N \mathbb{E}\left\{ \left(\mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{pp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \right)_{j,k} \right\}.$$

We define the $N \times N$ matrix $\mathbf{\Delta}_{i_1 i_2}^{m_1 m_2}(ff)$ by

$$\mathbf{\Delta}_{i_1 i_2}^{m_1 m_2}(ff) = -c_N \mathbb{E}\left\{ J_N^L \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{fp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(ff) \right\}$$
$$- c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{pp}}(I_L \otimes R)) \hat{\mathbf{A}}_{i_1 i_2}^{m_1 m_2}(pf) \right\}.$$

Dropping the indices i_1, i_2, m_1, m_2, we eventually obtain that

$$\mathbf{A_{ff}} = \mathbf{B_{ff}} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R)) \right\} \mathbf{A_{pf}} + \mathbf{\Delta_{ff}}.$$

Using similar calculations, it is possible to establish that :

$$\mathbf{A_{pf}} = \mathbf{B_{pf}} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{ff}}(I_L \otimes R)) \right\} \mathbf{A_{ff}} + \mathbf{\Delta_{pf}},$$
$$\mathbf{A_{fp}} = \mathbf{B_{fp}} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R)) \right\} \mathbf{A_{pp}} + \mathbf{\Delta_{fp}},$$
$$\mathbf{A_{pp}} = \mathbf{B_{pp}} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{ff}}(I_L \otimes R)) \right\} \mathbf{A_{fp}} + \mathbf{\Delta_{pp}},$$

where $\mathbf{\Delta_{pf}}$, $\mathbf{\Delta_{fp}}$, and $\mathbf{\Delta_{pp}}$ are defined as

$$\mathbf{\Delta_{pf}} = -c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{pf}}(I_L \otimes R)) J_N^{*L} \hat{\mathbf{A}}_{\mathbf{pf}} \right\} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{ff}}(I_L \otimes R)) \hat{\mathbf{A}}_{\mathbf{ff}} \right\},$$
$$\mathbf{\Delta_{fp}} = -c_N \mathbb{E}\left\{ J_N^L \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{fp}}(I_L \otimes R)) \hat{\mathbf{A}}_{\mathbf{fp}} \right\} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{pp}}(I_L \otimes R)) \hat{\mathbf{A}}_{\mathbf{pp}} \right\},$$
$$\mathbf{\Delta_{pp}} = -c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{pf}}(I_L \otimes R)) J_N^{*L} \hat{\mathbf{A}}_{\mathbf{pp}} \right\} - c_N \mathbb{E}\left\{ \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ_{\mathbf{ff}}(I_L \otimes R)) \hat{\mathbf{A}}_{\mathbf{fp}} \right\}.$$

By Lemma 3.6, matrices $\mathbb{E}\{\mathbf{Q_{ff}}\}$ and $\mathbb{E}\{\mathbf{Q_{pp}}\}$ are block diagonal. Therefore, matrices $\mathbb{E}\{\mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{ff}}(I_L \otimes R))\}$ and $\mathbb{E}\{\mathcal{T}_{N,L}^{(M)}(\mathbf{Q_{pp}}(I_L \otimes R))\}$ reduce to $\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{ff}}(I_L \otimes R)\}\, I_N$ and $\frac{1}{ML}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pp}}(I_L \otimes R)\}\, I_N$ respectively. As $\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{ff}}(I_L \otimes R)\} = \mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pp}}(I_L \otimes R)\}$ (see (3.36)), we eventually obtain that

$$\begin{pmatrix} I_N & \frac{c_N}{ML}\mathbb{E}\left\{\mathrm{Tr}\mathbf{Q_{pp}}(I_L \otimes R)\right\} I_N \\ \frac{c_N}{ML}\mathbb{E}\left\{\mathrm{Tr}\mathbf{Q_{pp}}(I_L \otimes R)\right\} I_N & I_N \end{pmatrix} \mathbf{A} = \mathbf{B} + \mathbf{\Delta}. \tag{3.54}$$

Using (3.42), this can be written as

$$\begin{pmatrix} I_N & c_N\, \boldsymbol{\alpha}_N\, I_N \\ c_N\, \boldsymbol{\alpha}_N\, I_N & I_N \end{pmatrix} \mathbf{A} = \mathbf{B} + \mathbf{\Delta}.$$

Lemma 3.4 implies that

$$1 - (c_N \boldsymbol{\alpha}(z))^2 \neq 0$$

if $z \in \mathbb{C}^+$. This implies that the matrix governing the linear system (3.54) is invertible for $z \in \mathbb{C}^+$. Matrix $\mathbf{H}$ given by

$$\mathbf{H} = \begin{pmatrix} I_N & c_N \boldsymbol{\alpha}(z)\, I_N \\ c_N \boldsymbol{\alpha}(z)\, I_N & I_N \end{pmatrix}^{-1}.$$

is thus well defined for each $z \in \mathbb{C}^+$. The blocks of $\mathbf{H}$ are of course given by

$$\mathbf{H_{pp}} = \mathbf{H_{ff}} = \frac{1}{1 - c_N^2 \boldsymbol{\alpha}(z)^2} I_N,$$
$$\mathbf{H_{pf}} = \mathbf{H_{fp}} = -\frac{c_N \boldsymbol{\alpha}(z)}{1 - c_N^2 \boldsymbol{\alpha}(z)^2} I_N.$$

(3.54) implies that $\mathbf{A} = \mathbf{HB} + \mathbf{H\Delta}$. (3.50) implies that we only need to evaluate matrices $\mathbf{A_{pf}}$ and $\mathbf{A_{fp}}$. We obtain that these matrices are given by

$$\mathbf{A_{pf}} = \mathbf{H_{pp}B_{pf}} + \mathbf{H_{pf}B_{ff}} + \mathbf{H_{pp}\Delta_{pf}} + \mathbf{H_{pf}\Delta_{ff}},$$
$$\mathbf{A_{fp}} = \mathbf{H_{fp}B_{pp}} + \mathbf{H_{ff}B_{fp}} + \mathbf{H_{fp}\Delta_{pp}} + \mathbf{H_{ff}\Delta_{fp}}.$$

This and definition (3.49) of matrix $\mathbf{A}_{i_1 i_2}^{m_1 m_2}$ lead immediately to

$$\left(\mathbb{E}\left\{\mathbf{Q}\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}\right\}\right)_{i_1 i_2}^{m_1 m_2} = \mathrm{Tr}\mathbf{A}_{i_1 i_2}^{m_1 m_2}(pf)\mathbf{1}_{i_2 \leq L} + \mathrm{Tr}\mathbf{A}_{i_1 i_2 - L}^{m_1 m_2}(fp)\mathbf{1}_{i_2 > L} =$$
$$\frac{1}{1 - c_N^2 \boldsymbol{\alpha}^2}\mathrm{Tr}\Big(\mathbf{B_{pf}} - c_N \boldsymbol{\alpha}\mathbf{B_{ff}} + \mathbf{\Delta_{pf}} - c_N \boldsymbol{\alpha}\mathbf{\Delta_{ff}}\Big)_{i_1 i_2}^{m_1 m_2}\mathbf{1}_{i_2 \leq L}$$
$$+ \frac{1}{1 - c_N^2 \boldsymbol{\alpha}^2}\mathrm{Tr}\Big(\mathbf{B_{fp}} - c_N \boldsymbol{\alpha}\mathbf{B_{pp}} + \mathbf{\Delta_{fp}} - c_N \boldsymbol{\alpha}\mathbf{\Delta_{pp}}\Big)_{i_1 i_2 - L}^{m_1 m_2}\mathbf{1}_{i_2 > L}.$$

It is easy to notice that $\mathrm{Tr}\,(\mathbf{B_{fp}})_{i_1 i_2}^{m_1 m_2} = \mathrm{Tr}\,(\mathbf{B_{pf}})_{i_1 i_2}^{m_1 m_2} = 0$, and $\mathrm{Tr}\,(\mathbf{B_{pp}})_{i_1 i_2}^{m_1 m_2} = \mathbb{E}\{(\mathbf{Q}\Pi_{ff}(I_2 L \otimes R))_{i_1 i_2 + L}^{m_1 m_2}\}$, $\mathrm{Tr}\,(\mathbf{B_{ff}})_{i_1 i_2}^{m_1 m_2} = \mathbb{E}\{(\mathbf{Q}\Pi_{pp}(I_2 L \otimes R))_{i_1 i_2}^{m_1 m_2}\}$, where $\Pi_{ff} = \begin{pmatrix} 0 & 0 \\ 0 & I_{ML} \end{pmatrix}$ and $\Pi_{pp} = \begin{pmatrix} I_{ML} & 0 \\ 0 & 0 \end{pmatrix}$. Hence, using that $\mathbb{E}\{\mathbf{Q_{pf}}\} = \mathbb{E}\{\mathbf{Q_{fp}}\} = 0$, we obtain that

$$\left(\mathbb{E}\left\{\mathbf{Q}\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}\right\}\right)_{i_1 i_2}^{m_1 m_2} = -\frac{c_N \boldsymbol{\alpha}}{1 - c_N^2 \boldsymbol{\alpha}^2}\Big(\mathbb{E}\{\mathbf{Q}\Pi_{pp}(I_{2L} \otimes R)\}$$
$$+ \mathbb{E}\{\mathbf{Q}\Pi_{ff}(I_{2L} \otimes R)\}\Big)_{i_1 i_2}^{m_1 m_2} + \mathcal{E}_{i_1 i_2}^{m_1 m_2} = -\frac{c_N \boldsymbol{\alpha}}{1 - c_N^2 \boldsymbol{\alpha}^2}\Big(\mathbb{E}\{\mathbf{Q}(I_{2L} \otimes R)\}\Big)_{i_1 i_2}^{m_1 m_2} + \mathcal{E}_{i_1 i_2}^{m_1 m_2},$$

where $\mathcal{E}_{i_1 i_2}^{m_1 m_2}$ represents the remaining terms depending on the entries of matrix $\mathbf{\Delta}_{i_1 i_2}^{m_1 m_2}$. Using the identity (3.47), we obtain that

$$z\mathbb{E}\{\mathbf{Q}\} + I_{2ML} = \mathbb{E}\left\{\mathbf{Q}\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}\right\} = -\frac{c_N \boldsymbol{\alpha}}{1 - c_N^2 \boldsymbol{\alpha}^2}\mathbb{E}\{\mathbf{Q}\}(I_{2L} \otimes R) + \mathcal{E}, \tag{3.55}$$

which immediately leads to

$$-\mathbb{E}\{\mathbf{Q}\}\left(\frac{c_N \boldsymbol{\alpha}}{1 - c_N^2 \boldsymbol{\alpha}^2}(I_{2L} \otimes R) + z\right) = I_{2ML} - \mathcal{E}$$

or, equivalently,

$$\mathbb{E}\{\mathbf{Q}\}\ (I_{2L} \otimes \mathbf{S})^{-1} = I_{2ML} - \mathcal{E},$$

where we recall that $\mathbf{S}$ is defined by (3.44). As $\mathbb{E}\{\mathbf{Q}\}$ is block diagonal, (3.55) implies that matrix $\mathcal{E}$ is also block diagonal, i.e. $\mathcal{E}_{\mathbf{fp}} = \mathcal{E}_{\mathbf{pf}} = 0$. Moreover, it holds that

$$\mathbb{E}\{\mathbf{Q}(z)\} = I_{2L} \otimes \mathbf{S}(z) - \mathcal{E}(z)\ (I_{2L} \otimes \mathbf{S}(z))\,. \tag{3.56}$$

This allows to evaluate $\mathbb{E}\{Q(z)\}$ by identification of the first diagonal blocks of the left and right hand sides of (3.56). We thus obtain immediately that

$$\mathbb{E}\{Q(z^2)\} = I_L \otimes S(z^2) - \mathcal{E}_{\mathbf{pp}}(z) \left(I_L \otimes S(z^2)\right) \tag{3.57}$$

for each $z \in \mathbb{C}^+$, where we recall that $S(z)$ is given by(3.43). Therefore, $\mathcal{E}_{\mathbf{pp}}(z)$ only depends on z^2. As the image of $\mathbb{C}^+$ by the transformation $z \to z^2$ is $\mathbb{C} - \mathbb{R}^+$, we obtain that $\mathcal{E}_{\mathbf{pp}}(z) = E(z^2)$ for some function E analytic in $\mathbb{C} - \mathbb{R}^+$. This discussion leads to

$$\mathbb{E}\{Q(z)\} = I_L \otimes S(z) - E(z) \ (I_L \otimes S(z)) \tag{3.58}$$

for each $z \in \mathbb{C} - \mathbb{R}^+$.
In the following, we prove (3.46). For this, we establish following result.

Proposition 3.1. *For each deterministic $ML \times ML$ sequence of matrices $(F_{1,N})_{N \geq 1}$ such that $\sup_{N\geq 1} \|F_{1,N}\| \leq \kappa$, then*

$$\left| \frac{1}{ML} \mathrm{Tr}(\mathcal{E}_{pp}(z)\, F_{1,N}) \right| \leq \kappa \, \frac{1}{N^2}\, P_1(|z^2|) P_2(\frac{1}{\mathrm{Im} z^2}) \tag{3.59}$$

holds for each $z \in \mathbb{C}^+$ for which $\mathrm{Im} z^2 > 0$, where P_1 and P_2 are 2 nice polynomials.

Proof. We define F_N as the $2ML \times 2ML$ matrix $F_N = \begin{pmatrix} F_{1,N} & 0 \\ 0 & 0 \end{pmatrix}$ and remark that $\frac{1}{ML}\mathrm{Tr}\mathcal{E}F = \frac{1}{ML}\mathrm{Tr}(\mathcal{E}_{\mathbf{pp}}(z)\, F_{1,N})$ can be written as

$$\frac{1}{ML}\mathrm{Tr}\mathcal{E}F = \frac{1}{1 - c^2 \boldsymbol{\alpha}^2} \sum_{\substack{i_1, i_2 \\ m_1, m_2}} \Big(\left(\mathrm{Tr}\boldsymbol{\Delta}_{i_1 i_2}^{m_1 m_2}(pf) - c\boldsymbol{\alpha}\mathrm{Tr}\boldsymbol{\Delta}_{i_1 i_2}^{m_1 m_2}(ff)\right) \mathbf{1}_{i_2 \leq L}$$
$$+ \left(\mathrm{Tr}\boldsymbol{\Delta}_{i_1 i_2 - L}^{m_1 m_2}(fp) - c\boldsymbol{\alpha}\mathrm{Tr}\boldsymbol{\Delta}_{i_1 i_2 - L}^{m_1 m_2}(pp)\right) \mathbf{1}_{i_2 > L} \Big) F_{i_2 i_1}^{m_2 m_1}. \tag{3.60}$$

As matrix F verifies $F_{i_2, i_1}^{m_2, m_1} = 0$ if $i_2 > L$, $\frac{1}{ML}\mathrm{Tr}\mathcal{E}F$ is reduced to the first term of the right hand side of (3.60) that we now evaluate.

$$\sum_{\substack{i_1, i_2 \\ m_1, m_2}} \mathrm{Tr}\boldsymbol{\Delta}_{i_1 i_2}^{m_1 m_2}(pf) F_{i_2 i_1}^{m_2 m_1} \mathbf{1}_{i_2 \leq L} = c \sum_{\substack{i_1, i_2 \\ m_1, m_2}} \sum_{j,k} \mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R))_{jk} \left(\mathbf{Q} \left(\begin{smallmatrix} w_{f,k} \\ 0 \end{smallmatrix}\right)\right)_{i_1}^{m_1}$$
$$\times \left(w_{f,j}^*\right)_{i_2}^{m_2} F_{i_2 i_1}^{m_2 m_1} + (\mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{pf}}^{\circ}(I_L \otimes R)) J_N^{*L})_{jk} \left(\mathbf{Q} \left(\begin{smallmatrix} 0 \\ w_{p,k} \end{smallmatrix}\right)\right)_{i_1}^{m_1} \left(w_{f,j}^*\right)_{i_2}^{m_2} F_{i_2 i_1}^{m_2 m_1} \Big\} \mathbf{1}_{i_2 \leq L}$$
$$= c\, \mathrm{Tr}\mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R)) \left(\begin{smallmatrix} W_f \\ 0 \end{smallmatrix}\right)^* F\mathbf{Q} \left(\begin{smallmatrix} W_f \\ 0 \end{smallmatrix}\right) + \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{pf}}^{\circ}(I_L \otimes R)) J_N^{*L} \left(\begin{smallmatrix} W_f \\ 0 \end{smallmatrix}\right)^* F\mathbf{Q} \left(\begin{smallmatrix} 0 \\ W_p \end{smallmatrix}\right) \Big\}$$
$$= c\, \mathrm{Tr}\mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R))\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{pf} W)$$
$$+ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{pf}}^{\circ}(I_L \otimes R)) J_N^{*L}\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{fp} W) \Big\}.$$

Similar calculations lead to the following expression of $\frac{1}{ML}\mathrm{Tr}\mathcal{E}F$:

$$\frac{1}{ML}\mathrm{Tr}\mathcal{E}F = \frac{c}{(1 - c_N^2 \boldsymbol{\alpha}^2)} \frac{1}{ML} \mathrm{Tr}\mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R))\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{pf} W)$$
$$+ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{pf}}^{\circ}(I_L \otimes R)) J_N^{*L}\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{fp} W) - c\boldsymbol{\alpha}\mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{pp}}^{\circ}(I_L \otimes R))\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{fp} W)$$
$$- c\boldsymbol{\alpha} J_N^L \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{fp}}^{\circ}(I_L \otimes R))\, (\Pi_{pf} W)^* \, F\mathbf{Q}\, (\Pi_{pf} W) \Big\}. \tag{3.61}$$

We now evaluate the right hand side of (3.61). The Schwartz inequality leads to

$$\left| \frac{1}{ML} \mathrm{Tr} \mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R)) \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big\} \right|$$

$$= \left| \sum_{l=-L+1}^{L-1} \mathbb{E}\Big\{ \tau^{(M)}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R))(l) \frac{1}{ML} \mathrm{Tr}\Big(J_N^{*\epsilon(l)} \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big) \Big\} \right|$$

$$= \left| \sum_{l=-L+1}^{L-1} \mathbb{E}\Big\{ \frac{1}{ML} \mathrm{Tr}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R)(J_L^{(l)} \otimes I_M)) \frac{1}{ML} \mathrm{Tr}\Big(J_N^{*\epsilon(l)} \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big)^{\circ} \Big\} \right|$$

$$\leq \sum_{l=-L+1}^{L-1} \mathbf{Var} \left\{ \frac{1}{ML} \mathrm{Tr}(\mathbf{Q}_{\mathbf{ff}}(I_L \otimes R)(J_L^{(l)} \otimes I_M)) \right\}^{1/2}$$

$$\times \, \mathbf{Var} \left\{ \frac{1}{ML} \mathrm{Tr}\Big(J_N^{*\epsilon(l)} \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big) \right\}^{1/2}.$$

Using Corollary 3.1, we obtain that

$$\mathbf{Var} \left\{ \frac{1}{ML} \mathrm{Tr}(\mathbf{Q}_{\mathbf{ff}}(I_L \otimes R)(J_L^{(l)} \otimes I_M)) \right\} \leq \frac{1}{N^2} P_1(|z^2|) P_2 \left(\frac{1}{\mathrm{Im} z^2} \right)$$

and that

$$\mathbf{Var} \left\{ \frac{1}{ML} \mathrm{Tr}\Big(J_N^{*\epsilon(l)} \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big) \right\} \leq \kappa^2 \frac{1}{N^2} P_1(|z^2|) P_2 \left(\frac{1}{\mathrm{Im} z^2} \right).$$

Since L does not grow with N, this implies immediately that

$$\left| \frac{1}{ML} \mathrm{Tr} \mathbb{E}\Big\{ \mathcal{T}_{N,L}^{M}(\mathbf{Q}_{\mathbf{ff}}^{\circ}(I_L \otimes R)) \, (\Pi_{pf} W)^{*} \, F \mathbf{Q} \, (\Pi_{pf} W) \, \Big\} \right| \leq \kappa \frac{1}{N^2} P_1(|z^2|) P_2 \left(\frac{1}{\mathrm{Im} z^2} \right)$$

holds. It can be shown similarly that the 3 other normalized traces can be upper bounded by the same kind of term. It remains to control the terms $\frac{1}{1-(c_N \, \boldsymbol{\alpha}_N)^2}$ and $\frac{\boldsymbol{\alpha}_N}{1-(c_N \, \boldsymbol{\alpha}_N)^2}$. For this, we use Lemma 3.5 for the choice $\beta_N(z) = \alpha_N(z)$. It is sufficient to verify that the measures $(\overline{\mu}_N)_{N \geq 1}$ associated to functions $(\alpha_N(z))_{N \geq 1}$ verify (3.29) and (3.30). For each N, it holds that

$$\int_0^{+\infty} d\,\overline{\mu}_N(\lambda) = \mathbb{E}\Big\{ \int_0^{+\infty} d\,\hat{\mu}_N(\lambda) \Big\} = \frac{1}{M} \mathrm{Tr} R_N$$

and

$$\int_0^{+\infty} \lambda \, d\,\overline{\mu}_N(\lambda) = \mathbb{E} \left(\int_0^{+\infty} \lambda \, d\,\hat{\mu}_N(\lambda) \right) = \mathbb{E} \left(\frac{1}{ML} \mathrm{Tr}((I_L \otimes R) W_f W_p^{*} W_p W_f^{*}) \right).$$

A straightforward calculation leads to $\mathbb{E}\Big\{ \frac{1}{ML} \mathrm{Tr}(W_f W_p^{*} W_p W_f^{*}) \Big\} = \frac{c_N}{M^2} \mathrm{Tr} R_N \mathrm{Tr} R_N^2$. Therefore, (3.32) implies that

$$\frac{1}{|1 - z(c_N \alpha_N(z))^2|} \leq P_1(|z|) P_2 \left(\frac{1}{\mathrm{Im} z} \right)$$

for each $z \in \mathbb{C}^+$, and if $z^2 \in \mathbb{C}^+$, it holds that

$$\frac{1}{|1 - z^2(c_N \alpha_N(z^2))^2|} \leq P_1(|z^2|) P_2 \left(\frac{1}{\mathrm{Im} z^2} \right).$$

As $\boldsymbol{\alpha}_N(z) = z \alpha_N(z^2)$, this is equivalent to

$$\frac{1}{1 - (c_N \, \boldsymbol{\alpha}_N)^2} \leq P_1(|z^2|) P_2 \left(\frac{1}{\mathrm{Im} z^2} \right).$$

Finally, we remark that $|\alpha_N(z)| \leq \frac{1}{M}\mathrm{Tr}R_N \frac{1}{\mathrm{Im}z} \leq b\,\frac{1}{\mathrm{Im}z}$ for each $z \in \mathbb{C}^+$. Therefore, if $z^2 \in \mathbb{C}^+$, it holds that $|\alpha_N(z^2)| \leq b\,\frac{1}{\mathrm{Im}z^2}$ and that $|\boldsymbol{\alpha}_N(z)| = |z||\alpha_N(z^2)|$ verifies

$$|\boldsymbol{\alpha}_N(z)| \leq b|z|\,\frac{1}{\mathrm{Im}z^2} \leq b(1+|z|^2)\,\frac{1}{\mathrm{Im}z^2}.$$

This completes the proof of Proposition 3.1. ■

Proposition 3.1 immediately leads to the following Corollary.

Corollary 3.2. *For each sequence $(F_N)_{N\geq 1}$ of deterministic $ML \times ML$ matrices such that $\sup_{N\geq 1}\|F_N\| \leq \kappa$ we have*

$$\left|\frac{1}{ML}\mathrm{Tr}\left[\left(\mathbb{E}\{Q_N(z)\} - I_L \otimes S_N(z)\right)F_N\right]\right| \leq \kappa\,\frac{1}{N^2}P_1(|z|)P_2\left(\frac{1}{\mathrm{Im}z^2}\right) \tag{3.62}$$

for each $z \in \mathbb{C}^+$. In particular, it holds that

$$\left|\frac{1}{ML}\mathrm{Tr}\left[\left(\mathbb{E}\{Q_N(z)\} - I_L \otimes S_N(z)\right)\right]\right| \leq \kappa\,\frac{1}{N^2}P_1(|z|)P_2\left(\frac{1}{\mathrm{Im}z^2}\right). \tag{3.63}$$

Proof. (3.57) implies that

$$\left|\frac{1}{ML}\mathrm{Tr}\left[\left(\mathbb{E}\{Q_N(z^2)\} - I_L \otimes S_N(z^2)\right)F_N\right]\right| = \left|\frac{1}{ML}\mathrm{Tr}\mathcal{E}_{\mathbf{pp}}(z)\left(I_L \otimes S_N(z^2)\right)F_N\right|$$

As $\mathcal{E}_{\mathbf{pp}}(z) = E(z^2)$ and $\|S_N(z^2)\| \leq \frac{1}{\mathrm{Im}z^2}$ if $z^2 \in \mathbb{C}^+$, the application of Proposition 3.1 to matrix $F_{1,N} = S_N(z^2)F_N$ implies that

$$\left|\frac{1}{ML}\mathrm{Tr}\left[\left(\mathbb{E}\{Q_N(z^2)\} - I_L \otimes S_N(z^2)\right)F_N\right]\right| \leq \kappa\frac{1}{N^2}P_1(|z^2|)P_2\left(\frac{1}{\mathrm{Im}z^2}\right)$$

for each z such that $z^2 \in \mathbb{C}^+$. Exchanging z^2 by z eventually establishes (3.62). This, in turn, completes the proof of Theorem 3.1.

3.5 Deterministic equivalent of $\mathbb{E}\{Q\}$

3.5.1 The canonical equation

Proposition 3.2. *If $z \in \mathbb{C}^+$, there exists a unique solution of the equation*

$$t_N(z) = \frac{1}{M}\mathrm{Tr}R_N\left(-zI_M - \frac{zc_N t_N(z)}{1 - zc_N^2 t_N^2(z)}R_N\right)^{-1} \tag{3.64}$$

satisfying $t_N(z) \in \mathbb{C}^+$ and $zt_N(z) \in \mathbb{C}^+$. Function $z \to t_N(z)$ is an element of $\mathcal{S}(\mathbb{R}^+)$, and the associated positive measure, denoted by μ_N, verifies

$$\mu_N(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}R_N, \quad \int_{\mathbb{R}^+}\lambda\, d\,\mu_N(\lambda) = c_N\,\frac{1}{M}\mathrm{Tr}R_N\,\frac{1}{M}\mathrm{Tr}R_N^2. \tag{3.65}$$

Moreover, it exists nice constants β and κ such that

$$\frac{1}{\left|1 - z\left(c_N\, t_N(z)\right)^2\right|} \leq \frac{\kappa\left(\beta^2 + |z|^2\right)^2}{\left(\mathrm{Im}z\right)^3} \tag{3.66}$$

for each N. Finally, the $M \times M$ valued function $T_N(z)$ defined by

$$T_N(z) = -\left(zI_M + \frac{zc_N t_N(z)}{1 - zc_N^2 t_N^2(z)}R_N\right)^{-1} \tag{3.67}$$

belongs to $\mathcal{S}_M(\mathbb{R}^+)$. The associated $M \times M$ positive matrix-valued measure, denoted ν_N^T, verifies

$$\nu_N^T(\mathbb{R}^+) = I_M \tag{3.68}$$

as well as

$$\mu_N = \frac{1}{M}\mathrm{Tr}R_N\nu_N^T. \tag{3.69}$$

Proof. As N is assumed to be fixed in the statement of the Proposition, we omit to mention that $t_N, T_N, \mu_N, \dots$ depend on N in the course of the proof. We first prove the existence of a solution such that $z \to t(z)$ is an element of $\mathcal{S}(\mathbb{R}^+)$. For this, we use the classical fixed point equation scheme. We define $t_0(z) = -\frac{1}{z}$, which is of course an element of $\mathcal{S}(\mathbb{R}^+)$, and generate sequence $(t_n(z))_{n\geq 1}$ by the formula

$$t_{n+1}(z) = \frac{1}{M}\mathrm{Tr}R\left(-zI_M - \frac{zct_n(z)}{1-zc^2t_n^2(z)}R\right)^{-1}.$$

We establish by induction that for each n, $t_n \in \mathcal{S}(\mathbb{R}^+)$, and that its associated measure μ_n verifies $\mu_n(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}R$ and

$$\int_0^{+\infty} \lambda\mu_n(d\lambda) = c\frac{1}{M}\mathrm{Tr}(R)\frac{1}{M}\mathrm{Tr}(R^2). \tag{3.70}$$

Thanks to (2.2), this last property will imply that sequence $(\mu_n)_{n\geq 1}$ is tight. We assume that t_n indeed satisfies the above conditions, and prove that $t_{n+1}(z)$ also meets these requirements. Lemma 3.4 implies that function $T_n(z) = \left(-zI_M - \frac{zct_n(z)}{1-zc^2t_n^2(z)}R\right)^{-1}$ is an element of $\mathcal{S}_M(\mathbb{R}^+)$. According to Proposition 2.1, to prove that $t_{n+1}(z) \in \mathcal{S}(\mathbb{R}^+)$, we need to check that $\mathrm{Im}t_{n+1}(z), \mathrm{Im}zt_{n+1}(z) > 0$ if $z \in \mathbb{C}^+$, as well as that $\lim_{y\to+\infty} iyt_{n+1}(iy)$ exists. As $T_n \in \mathcal{S}_M(\mathbb{R}^+)$ and $t_{n+1}(z) = \frac{1}{M}\mathrm{Tr}RT_n(z)$, it is clear that $\mathrm{Im}t_{n+1}(z), \mathrm{Im}zt_{n+1}(z) > 0$. Finally, it holds that

$$-iyt_{n+1}(iy) = \frac{1}{M}\mathrm{Tr}R\left(I_M + \frac{ciyt_n(iy)}{iy - (ciyt_n(iy))^2}R\right)^{-1}.$$

Since $t_n(z)$ is a Stieltjes transform we have $-iyt_n(iy) \to \mu_n(\mathbb{R}^+)$, which implies that $-iyt_{n+1}(iy) \to \frac{1}{M}\mathrm{Tr}R$, i.e. that $\mu_{n+1}(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}R$.
We finally check that μ_{n+1} satisfies (3.70). For this, we follow [18].

$$\int_0^{+\infty} \lambda\mu_{n+1}(d\lambda) = \lim_{y\to+\infty} \mathrm{Re}\left(-iy(iy\frac{1}{M}\mathrm{Tr}RT_n(iy) + \frac{1}{M}\mathrm{Tr}R)\right).$$

We can express T_n as

$$T_n = -\frac{1}{z}\left(I_M + \frac{ct_n}{1-zc^2t_n^2}R\right)^{-1} = -\frac{1}{z} + \frac{R}{z}\frac{ct_n}{1-zc^2t_n^2} - \left(\frac{ct_n}{1-zc^2t_n^2}\right)^2 R^2T_n,$$

from which it follows that

$$-z\left(\frac{1}{M}\mathrm{Tr}(zRT_n(z)) + \frac{1}{M}\mathrm{Tr}R\right) = -\frac{czt_n}{1-zc^2t_n^2}\frac{1}{M}\mathrm{Tr}R^2 + \left(\frac{czt_n}{1-zc^2t_n^2}\right)^2\frac{1}{M}\mathrm{Tr}R^3T_n.$$

Since $-iyt_n(iy) \to \frac{1}{M}\mathrm{Tr}R$ and $t_n(iy) \to 0$ we can conclude that $-iy(iy\frac{1}{M}\mathrm{Tr}RT_n(iy) + \frac{1}{M}\mathrm{Tr}R) \to \frac{c}{M^2}\mathrm{Tr}R\mathrm{Tr}R^2$ as expected.

We now prove that sequence t_n converges towards a function $t \in \mathcal{S}(\mathbb{R}^+)$ verifying equation (3.64). For this we evaluate $\theta_n = t_{n+1} - t_n$

$$\begin{aligned}\theta_n &= \frac{1}{M}\mathrm{Tr}R(T_n - T_{n-1}) = \frac{1}{M}\mathrm{Tr}RT_n\frac{zc(t_n - t_{n-1})(1+zc^2t_nt_{n-1})}{(1-zc^2t_n^2)(1-zc^2t_{n-1}^2)}RT_{n-1}\\ &= \theta_{n-1}\frac{zc(1+zc^2t_nt_{n-1})}{(1-zc^2t_n^2)(1-zc^2t_{n-1}^2)}\frac{1}{M}\mathrm{Tr}RT_nRT_{n-1}.\end{aligned}$$

We denote by $f_n(z)$ the term defined by

$$f_n(z) = \frac{zc(1+zc^2t_nt_{n-1})}{(1-zc^2t_n^2)(1-zc^2t_{n-1}^2)}\frac{1}{M}\mathrm{Tr}RT_nRT_{n-1}. \tag{3.71}$$

Lemma 3.4 implies that $\|T_k\| \leq \frac{1}{\mathrm{Im} z}$ and that $|t_k| \leq \frac{b}{\mathrm{Im} z}$ for each $k \geq 1$ and each $z \in \mathbb{C}^+$. Therefore, it holds that

$$\left| zc(1 + zc^2 t_n t_{n-1}) \frac{1}{M} \mathrm{Tr} R T_n R T_{n-1} \right| \leq \kappa \left(\frac{|z|}{(\mathrm{Im} z)^2} \left(1 + \frac{|z|}{(\mathrm{Im} z)^2} \right) \right).$$

Moreover, it is clear that for each k, $|1 - zc^2 t_k^2| \geq (1 - c^2 \frac{|z|}{(\mathrm{Im} z)^2})$. For each $\epsilon > 0$ small enough, we consider the domain $\mathcal{D}_\epsilon$ defined by

$$\mathcal{D}_\epsilon = \{ z \in \mathbb{C}^+, \frac{|z|}{(\mathrm{Im} z)^2} < \epsilon \}. \tag{3.72}$$

Then, for $z \in \mathcal{D}_\epsilon$, it holds that

$$\frac{1}{|1 - zc^2 t_n^2|} \frac{1}{|1 - zc^2 t_{n-1}^2|} \leq \frac{1}{(1 - c^2 \epsilon)^2}$$

and that

$$|f_n(z)| \leq \frac{\kappa}{(1 - c^2 \epsilon)^2} \left(\epsilon + \epsilon^2 \right).$$

We choose ϵ in such a way that $\frac{\kappa}{(1-c^2\epsilon)^2} \left(\epsilon + \epsilon^2 \right) < 1/2$. Then, for each $z \in \mathcal{D}_\epsilon$, it holds that

$$|\theta_n| \leq \frac{1}{2} |\theta_{n-1}|.$$

Therefore, for each z in $\mathcal{D}_\epsilon$, $(t_n(z))_{n \geq 1}$ is a Cauchy sequence. We denote by $t(z)$ its limit. $(t_n(z))_{n \geq 1}$ is uniformly bounded on every compact set of $\mathbb{C} - \mathbb{R}^+$. This implies that $(t_n(z))_{n \geq 1}$ is a normal family on $\mathbb{C} - \mathbb{R}^+$. We consider a converging subsequence extracted from $(t_n(z))_{n \geq 1}$. The corresponding limit $t_*(z)$ is analytic over $\mathbb{C} - \mathbb{R}^+$. If $z \in \mathcal{D}_\epsilon$, $t_*(z)$ must be equal to $t(z)$. Therefore, the limits of all converging subsequences extracted from $(t_n(z))_{n \geq 1}$ must coincide on $\mathcal{D}_\epsilon$, and therefore on $\mathbb{C} - \mathbb{R}^+$. This implies that $t_n(z)$ converges uniformly on each compact subset towards a function which is analytic $\mathbb{C} - \mathbb{R}^+$, and that we also denote by $t(z)$. It is clear that $t(z)$ verifies (3.64) and that $t \in \mathcal{S}(\mathbb{R}^+)$ and verifies (3.65). Moroever, Lemma 3.4 implies that $T \in \mathcal{S}_M(\mathbb{R}^+)$, while (3.69) and (3.68) are obtained immediately.

As (3.65) holds, (3.66) is a consequence of the application of Lemma 3.5 to the function $\beta_N(z) = t_N(z)$.

We now prove that if $z \in \mathbb{C}^+$ and $t_1(z)$ and $t_2(z)$ are 2 solutions of (3.64) such that $t_i(z)$ and $zt_i(z)$ belong to $\mathbb{C}^+$, $i = 1, 2$, then $t_1(z) = t_2(z)$. In order to prove this, we first establish the following useful Lemma.

Lemma 3.7. *If $z \in \mathbb{C}^+$ and if $t(z)$ verifies the conditions of Proposition 3.2, then, it holds that*

$$1 - u(z) > 0 \tag{3.73}$$

and

$$\det \left(\mathbf{I} - \mathbf{D} \right) > 0, \tag{3.74}$$

where

$$\mathbf{D} = \begin{pmatrix} u(z) & v(z) \\ |z|^2 v(z) & u(z) \end{pmatrix}, \tag{3.75}$$

$$u(z) = c \, \frac{|czt(z)|^2 \, \frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^* R)}{|1 - z(ct(z))^2|^2}, \tag{3.76}$$

$$v(z) = c \, \frac{\frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^* R)}{|1 - z(ct(z))^2|^2}. \tag{3.77}$$

Proof. Using the equation $t(z) = \frac{1}{M} \mathrm{Tr} RT(z)$, we obtain immediately after some algebra that

$$\begin{pmatrix} \frac{\mathrm{Im}(t(z))}{\mathrm{Im}(z)} \\ \frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)} \end{pmatrix} = \mathbf{D} \begin{pmatrix} \frac{\mathrm{Im}(t(z))}{\mathrm{Im}(z)} \\ \frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)} \end{pmatrix} + \begin{pmatrix} \frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^*) \\ 0 \end{pmatrix}. \tag{3.78}$$

The first component of (3.78) implies that

$$(1 - u(z)) \frac{\mathrm{Im}(t(z))}{\mathrm{Im}(z)} = v(z) \frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)} + \frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^*).$$

Therefore, it holds that $(1 - u(z)) > 0$. Plugging the equality

$$\frac{\mathrm{Im}(t(z))}{\mathrm{Im}(z)} = \frac{v(z)}{1 - u(z)} \frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)} + \frac{1}{1 - u(z)} \frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^*)$$

into the second component of (3.78) leads to

$$\left(1 - u(z) - \frac{|z|^2 v^2(z)}{1 - u(z)}\right) \frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)} = \frac{|z|^2 v(z)}{1 - u(z)} \frac{1}{M} \mathrm{Tr}(RT(z)(T(z))^*) > 0$$

and to (3.74).

To complete the proof of the uniqueness, we assume that equation (3.64) has 2 solutions $t_1(z)$ and $t_2(z)$ such that $t_i(z)$ and $zt_i(z)$ belong to $\mathbb{C}^+$ for $i = 1, 2$. The proof of Lemma 3.4 (see in particular (3.26)) implies that for $i = 1, 2$, then $1 - z(ct_i(z))^2 \neq 0$ and matrix $-zI - \frac{zct_i(z)}{1 - zc^2 t_i^2(z)} R$ is invertible. We denote by $T_1(z)$ and $T_2(z)$ the matrices defined by (3.67) when $t(z) = t_1(z)$ and $t(z) = t_2(z)$ respectively. $u_i(z)$ and $v_i(z)$, $i = 1, 2$, are defined similarly from (3.76) and (3.77) when $t(z) = t_1(z)$ and $t(z) = t_2(z)$. Using that $t_i(z) = \frac{1}{M} \mathrm{Tr}(RT_i(z))$ for $i = 1, 2$, we obtain immediately that

$$t_1(z) - t_2(z) = (u_{1,2}(z) + zv_{1,2}(z)) \, (t_1(z) - t_2(z)),$$

where

$$u_{1,2}(z) = c \, \frac{czt_1(z)czt_2(z) \, \frac{1}{M} \mathrm{Tr}(RT_1(z)RT_2(z))}{(1 - z(ct_1(z))^2) \, (1 - z(ct_2(z))^2)} \tag{3.79}$$

and

$$v_{1,2}(z) = c \, \frac{\frac{1}{M} \mathrm{Tr}(RT_1(z)RT_2(z))}{(1 - z(ct_1(z))^2) \, (1 - z(ct_2(z))^2)}. \tag{3.80}$$

In order to prove that $t_1(z) = t_2(z)$, it is sufficient establish that $1 - u_{1,2}(z) - zv_{1,2}(z) \neq 0$. For this, we prove the following inequality :

$$|1 - u_{1,2}(z) - zv_{1,2}(z)| > \sqrt{(1 - u_1(z)) - |z| v_1(z)} \sqrt{(1 - u_2(z)) - |z| v_2(z)} \tag{3.81}$$

which, by Lemma 3.7, implies $1 - u_{1,2}(z) - zv_{1,2}(z)) \neq 0$. For this, we remark that the Schwartz inequality leads to $|u_{1,2}(z)| \leq \sqrt{u_1(z)} \sqrt{u_2(z)}$ and $|v_{1,2}(z)| \leq \sqrt{v_1(z)} \sqrt{v_2(z)}$. Therefore,

$$|1 - u_{1,2}(z) - zv_{1,2}(z)| \geq 1 - \sqrt{u_1(z)} \sqrt{u_2(z)} - \sqrt{|z| v_1(z)} \sqrt{|z| v_2(z)}.$$

We now use the inequality

$$\sqrt{ab} - \sqrt{cd} \geq \sqrt{a - c} \sqrt{b - d}, \tag{3.82}$$

where a, b, c, d are positive real numbers such that $a \geq c$ and $b \geq d$. (3.82) for $a = b = 1$ and $c = u_1(z)$, $d = u_2(z)$ implies that $1 - \sqrt{u_1(z)} \sqrt{u_2(z)} \geq \sqrt{1 - u_1(z)} \sqrt{1 - u_2(z)}$. Therefore, it holds that

$$|1 - u_{1,2}(z) - zv_{1,2}(z)| \geq \sqrt{1 - u_1(z)} \sqrt{1 - u_2(z)} - \sqrt{|z| v_1(z)} \sqrt{|z| v_2(z)}.$$

(3.82) for $a = 1 - u_1(z)$, $b = 1 - u_2(z)$, $c = |z| v_1(z)$ and $d = |z| v_2(z)$ eventually leads to (3.81). This completes the proof of the uniqueness of the solution of (3.64) and Proposition 3.2. ■

Remark 3.1. *(3.73) and (3.74) are still valid if z belongs to $\mathbb{R}^{-*}$. To check this, it is sufficient to remark if $z = x \in \mathbb{R}^{-*}$, the fundamental equation (3.78) is still valid, but $\frac{\mathrm{Im}(t(z))}{\mathrm{Im}(z)}$ and $\frac{\mathrm{Im}(zt(z))}{\mathrm{Im}(z)}$ have to be replaced by $t'(x)$ and $(xt(x))'$ where $'$ denotes the differentiation operator w.r.t. x. The same conclusions are obtained because $t'(x) > 0$ and $(xt(x))' > 0$ if $x \in \mathbb{R}^{-*}$.*

3.5.2 Convergence

In this paragraph, we establish that the empirical eigenvalue distribution $\hat{\nu}_N$ of matrix $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ has almost surely the same deterministic behaviour than the probability measure ν_N defined by

$$\nu_N = \frac{1}{M}\mathrm{Tr}\nu_N^T, \tag{3.83}$$

where we recall that ν_N^T represents the positive matrix valued measure associated to $T_N(z)$. For this, we first establish the following Proposition.

Proposition 3.3. *For each sequence $(F_N)_{N\geq 1}$ of deterministic $ML \times ML$ matrices such that $\sup_{N\geq 1}\|F_N\| \leq \kappa$, then,*

$$\frac{1}{ML}\mathrm{Tr}\left[\left(\mathbb{E}\{Q_N(z)\} - I_L\otimes T_N(z)\right)F_N\right] \to 0 \tag{3.84}$$

holds for each $z \in \mathbb{C} - \mathbb{R}^+$.

Proof. Corollary 3.2 implies that

$$\frac{1}{ML}\mathrm{Tr}(\mathbb{E}\{Q_N\} - (I_L\otimes S_N))F_N = \mathcal{O}\left(\frac{1}{N^2}\right).$$

We have therefore to show that $\frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S_N - T_N)\right)F_N \to 0$. It is easy to check that

$$\begin{aligned}\frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S-T)\right)F &= \frac{1}{ML}\mathrm{Tr}(I_L\otimes S)\left(\frac{zc_N\alpha}{1-zc_N^2\alpha^2} - \frac{zc_Nt}{1-zc_N^2t^2}\right)(I_L\otimes RT)F \\ &= \frac{zc_N(\alpha-t)(1+zc_N^2\alpha t)}{(1-zc_N^2\alpha^2)(1-zc_N^2t^2)}\frac{1}{ML}\mathrm{Tr}(I_L\otimes SRT)F. \end{aligned}\tag{3.85}$$

We express $\alpha - t$ as $\alpha - \frac{1}{M}\mathrm{Tr}RS + \frac{1}{M}\mathrm{Tr}R(S-T)$, and deduce from (3.85) that

$$\begin{aligned}\frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S-T)\right)F &= \left(\alpha - \frac{1}{M}\mathrm{Tr}RS\right)\frac{zc_N(1+zc_N^2\alpha t)}{(1-zc_N^2\alpha^2)(1-zc_N^2t^2)} \\ &\times \frac{1}{ML}\mathrm{Tr}(I_L\otimes SRT)F + \frac{1}{M}\mathrm{Tr}R(S-T)\,\frac{zc_N(1+zc_N^2\alpha t)}{(1-zc_N^2\alpha^2)(1-zc_N^2t^2)}\frac{1}{ML}\mathrm{Tr}(I_L\otimes SRT)F. \end{aligned}\tag{3.86}$$

(3.62) implies that $\alpha - \frac{1}{M}\mathrm{Tr}RS = \mathcal{O}_z(\frac{1}{N^2})$. Therefore, in order to establish (3.84), it is sufficient to prove that $\frac{1}{M}\mathrm{Tr}R(S-T) \to 0$. For this, we take $F = I_L\otimes R$ in (3.86) and get that

$$\frac{1}{M}\mathrm{Tr}R(S(z)-T(z)) = f_N(z)\,\frac{1}{M}\mathrm{Tr}R(S(z)-T(z)) + \mathcal{O}_z(\frac{1}{N^2}) \tag{3.87}$$

where $f_N(z)$ is defined by

$$f_N(z) = \frac{zc_N(1+zc_N^2\alpha t)}{(1-zc_N^2\alpha^2)(1-zc_N^2t^2)}\frac{1}{M}\mathrm{Tr}(RS(z)RT(z)).$$

$f_N(z)$ is similar to the term defined in (3.71). Using the arguments of the proof of Proposition 3.2, we obtain that it is possible to find $\epsilon > 0$ for which, $\sup_{N\geq N_0}|f_N(z)| < \frac{1}{2}$ for each $z \in \mathcal{D}_\epsilon$ for some large enough integer N_0. We recall that $\mathcal{D}_\epsilon$ is defined by (3.72). We therefore deduce from (3.87) that $\frac{1}{M}\mathrm{Tr}R(S(z) - T(z)) \to 0$ and $\frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S(z)-T(z))\right)F$ converge towards 0 for each $z \in \mathcal{D}_\epsilon$. As functions $z \to \frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S_N(z)-T_N(z))\right)F_N$ are holomorphic on $\mathbb{C}-\mathbb{R}^+$ and are uniformly bounded on each compact subset of $\mathbb{C}-\mathbb{R}^+$, we deduce from Montel's theorem that $\frac{1}{ML}\mathrm{Tr}\left(I_L\otimes(S_N(z)-T_N(z))\right)F_N$ converges towards 0 for each $z\in\mathbb{C}-\mathbb{R}^+$. ■

We deduce the following Corollary.

Corollary 3.3. *The empirical eigenvalue distribution $\hat{\nu}_N$ of $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ verifies*

$$\hat{\nu}_N - \nu_N \to 0 \tag{3.88}$$

weakly almost surely.

Proof. Proposition 3.3 implies that $\mathbb{E}\{\frac{1}{ML}\text{Tr}Q_N(z)\} - \frac{1}{M}\text{Tr}(T_N(z)) \to 0$ for each $z \in \mathbb{C} - \mathbb{R}^+$. The Poincaré-Nash inequality and the Borel Cantelli Lemma imply that $\frac{1}{ML}\text{Tr}(Q_N(z)) - \mathbb{E}\{\frac{1}{ML}\text{Tr}Q_N(z)\} \to 0$ a.s. for each $z \in \mathbb{C} - \mathbb{R}^+$. Therefore, it holds that

$$\frac{1}{ML}\text{Tr}(Q_N(z)) - \frac{1}{M}\text{Tr}(T_N(z)) \to a.s. \tag{3.89}$$

for each $z \in \mathbb{C} - \mathbb{R}^+$. Corollary 2.7 of [18] implies that $\hat{\nu}_N - \nu_N \to 0$ weakly almost surely provided we verify that $(\hat{\nu}_N)_{N\geq 1}$ is almost surely tight and that $(\nu_N)_{N\geq 1}$ is tight. It is clear that

$$\int_{\mathbb{R}^+} \lambda \, d\, \hat{\nu}_N(\lambda) = \frac{1}{ML}\text{Tr}W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^* \leq \|W_N\|^4,$$

where we recall that

$$W_N = \begin{pmatrix} W_{p,N} \\ W_{f,N} \end{pmatrix}.$$

It holds that $\|W_N\| \leq \sqrt{b}\,\|W_{iid,N}\|$ where $W_{iid,N}$ is defined by (3.5). As $\|W_{iid,N}\| \to (1+\sqrt{c_*})$ almost surely (see [32]), we obtain that $\frac{1}{ML}\text{Tr}W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ is almost surely bounded for N large enough. This implies that $(\hat{\nu}_N)_{N\geq 1}$ is almost surely tight. As for sequence $(\nu_N)_{N\geq 1}$, we have shown that $\sup_N \int_{\mathbb{R}^+} \lambda \, d\, \mu_N(\lambda) < +\infty$. As $\mu_N = \frac{1}{M}\text{Tr}R_N\nu_N^T$, the condition $R_N > aI$ for each N leads to

$$\int_{\mathbb{R}^+} \lambda \, d\, \mu_N(\lambda) \geq a \int_{\mathbb{R}^+} \lambda \, d\, \nu_N(\lambda).$$

Therefore, it holds that $\sup_N \int_{\mathbb{R}^+} \lambda \, d\, \nu_N(\lambda) < +\infty$, a condition which implies that $(\nu_N)_{N\geq 1}$ is tight. ■

3.6 Detailed study of ν_N.

In this section, we study the properties of ν_N. (2.2) implies that μ_N and ν_N are absolutely continuous one with respect each other. Hence, they share the same properties, and the same support denoted $\mathcal{S}_N$ in the following. We thus study μ_N and deduce the corresponding results related to ν_N. As in the context of other models, μ_N can be characterized by studying the Stieltjes transform $t_N(z)$ near the real axis. In the following, we denote by $\overline{M}$ the number of distinct eigenvalues $(\overline{\lambda}_{l,N})_{l=1,\ldots,\overline{M}}$ of R_N arranged in the decreasing order, and by $(m_{l,N})_{l=1,\ldots,\overline{M}}$ their multiplicities. It of course holds that $\sum_{l=1}^{\bar{M}} m_{l,N} = M$.

3.6.1 Properties of $t(z)$ near the real axis.

In this paragraph, we establish that if $x_0 \in \mathbb{R}^{+*}$, then, $\lim_{z\to x_0, z\in\mathbb{C}^+} t(z)$ exists and is finite. It will be denoted by $t(x_0)$ in order to simplify the notations. Moreover, when $c \leq 1$, $\lim_{z\to 0, z\in\mathbb{C}^+\cup\mathbb{R}^*} |t(z)| = +\infty$, and $\lim_{z\to 0, z\in\mathbb{C}^+\cup\mathbb{R}^*} zt(z) = 0$. The results of [43] will imply that measure μ_N is absolutely continuous w.r.t. the Lebesgue measure, and that the corresponding density is equal to $\frac{1}{\pi}\text{Im}(t(x))$ for each $x \in \mathbb{R}^{+*}$. When $c > 1$, a Dirac mass appears at 0.

We first address the case where $x_0 \neq 0$, and, in order to establish the existence of $\lim_{z\to x_0, z\in\mathbb{C}^+} t(z)$, we prove the following properties :

- If $(z_n)_{n\geq 1}$ is a sequence of $\mathbb{C}^+$ converging towards x_0, then $|t(z_n)|_{n\geq 1}$ is bounded
- If $(z_{1,n})_{n\geq 1}$ and $(z_{2,n})_{n\geq 1}$ are two sequences of $\mathbb{C}^+$ converging towards x_0 and verifying $\lim_{z_{i,n}\to x_0} = t_i$ for $i = 1, 2$, then $t_1 = t_2$.

Lemma 3.8. *If $x_0 \in \mathbb{R}^{+*}$, and if $(z_n)_{n\geq 1}$ is a sequence of $\mathbb{C}^+$ such that $\lim_{n\to+\infty} z_n = x_0$, then the set $|t(z_n)|_{n\geq 1}$ is bounded.*

Proof. We assume that $|t(z_n)| \to +\infty$. Equation (3.64) can be written as

$$t(z_n) = \frac{1}{M} \sum_{l=1}^{\overline{M}} \frac{m_l \, \overline{\lambda}_l}{-z_n(1 + \frac{ct(z_n)\overline{\lambda}_l}{1-z\,(ct(z_n))^2})}. \tag{3.90}$$

As $x_0 \neq 0$, the condition $|t(z_n)| \to +\infty$ implies that it exists l_0 for which

$$1 + \frac{ct(z_n)\overline{\lambda}_{l_0}}{1 - z\,(ct(z_n))^2} \to 0$$

or equivalently

$$z_n ct(z_n) - \frac{1}{ct(z_n)} \to \overline{\lambda}_{l_0}.$$

As $|t(z_n)| \to +\infty$, it holds that $z_n ct(z_n) \to \overline{\lambda}_{l_0}$, a contradiction because $|z_n ct(z_n)| \to +\infty$. ■

Lemma 3.9. *Consider $(z_{1,n})_{n\geq 1}$ and $(z_{2,n})_{n\geq 1}$ two sequences of $\mathbb{C}^+$ converging towards $x_0 \in \mathbb{R}^{+*}$ and verifying $\lim_{z_{i,n}\to x_0} t(z_{i,n}) = t_i$ for $i = 1, 2$. Then, it holds that $t_1 = t_2$.*

Proof. The statement of the Lemma is obvious if x_0 does not belong to $\mathcal{S}$. Therefore, we assume that $x_0 \in \mathcal{S} - \{0\}$. We first observe that if $\lim_{n\to+\infty} z_n = x_0$ ($z_n \in \mathbb{C}^+$) and $t(z_n) \to t_0$, then

$$1 - x_0\,(ct_0)^2 \neq 0, \tag{3.91}$$

$$1 + \frac{ct_0\,\overline{\lambda}_l}{1 - x_0\,(ct_0)^2} \neq 0, \; l = 1, \ldots, \overline{M}. \tag{3.92}$$

Indeed, if (3.91) does not hold, Eq. (3.90) leads to $t_0 = 0$, a contradiction because $1 - x_0\,(ct_0)^2$ was assumed equal to 0. Similarly, if (3.92) does not hold, the limit of $t(z_n)$ cannot be finite. Therefore, matrix T_0 defined by

$$T_0 = -\left(x_0 \left[I + \frac{ct_0}{1 - x_0\,(ct_0)^2}\,R\right]\right)^{-1} \tag{3.93}$$

is well defined, and it holds that $T(z_n) \to T_0$ and that $t_0 = \frac{1}{M}\mathrm{Tr}RT_0$. In particular, for $i = 1, 2$, $T(z_{i,n}) \to T_i$ where T_i is defined by (3.93) when $t_0 = t_i$, $i = 1, 2$, and $t_i = \frac{1}{M}\mathrm{Tr}RT_i$. Using the equation (3.64) for $z = z_{i,n}$, we obtain immediately that

$$\begin{pmatrix} t(z_{1,n}) - t(z_{2,n}) \\ z_{1,n}t(z_{1,n}) - z_{2,n}t(z_{2,n}) \end{pmatrix} = \begin{pmatrix} u_0(z_{1,n}, z_{2,n}) & v_0(z_{1,n}, z_{2,n}) \\ z_{1,n}z_{2,n}v_0(z_{1,n}, z_{2,n}) & u_0(z_{1,n}, z_{2,n}) \end{pmatrix}$$
$$\times \begin{pmatrix} t(z_{1,n}) - t(z_{2,n}) \\ z_{1,n}t(z_{1,n}) - z_{2,n}t(z_{2,n}) \end{pmatrix} + \begin{pmatrix} (z_{1,n} - z_{2,n})\,\frac{1}{M}\mathrm{Tr}T(z_{1,n})RT(z_{2,n}) \\ 0 \end{pmatrix}, \tag{3.94}$$

where $u_0(z_1, z_2)$ and $v_0(z_1, z_2)$ are defined by

$$u_0(z_1, z_2) = c\,\frac{cz_1t(z_1)cz_2t(z_2)\,\frac{1}{M}\mathrm{Tr}(RT(z_1)RT(z_2))}{(1 - z_1(ct(z_1))^2)\,(1 - z_2(ct(z_2))^2)} \tag{3.95}$$

and

$$v_0(z_1, z_2) = c\,\frac{\frac{1}{M}\mathrm{Tr}(RT(z_1)RT(z_2))}{(1 - z_1(ct(z_1))^2)\,(1 - z_2(ct(z_2))^2)} \tag{3.96}$$

for $z_i \in \mathbb{C}^+$, $i = 1, 2$. Taking the limit, we obtain that

$$\begin{pmatrix} t_1 - t_2 \\ x_0(t_1 - t_2) \end{pmatrix} = \begin{pmatrix} u_0(x_0, x_0) & v_0(x_0, x_0) \\ x_0^2 v_0(x_0, x_0) & u_0(x_0, x_0) \end{pmatrix} \begin{pmatrix} t_1 - t_2 \\ x_0(t_1 - t_2) \end{pmatrix},$$

where $u_0(x_0, x_0)$ and $v_0(x_0, x_0)$ are defined by replacing $z_i, t(z_i), T(z_i)$ by x_0, t_i, T_i in (3.95, 3.96) for $i = 1, 2$. If the determinant $(1 - u_0(x_0, x_0))^2 - x_0^2 v_0(x_0, x_0)^2 \neq 0$ of the above linear system is non zero, it of course holds that $t_1 = t_2$.

We now consider the case where $(1-u_0(x_0,x_0))^2 - x_0^2 v_0(x_0,x_0)^2 = 0$. We denote by $u_i(x_0)$ and $v_i(x_0)$, $i = 1,2$ the limits of $u(z_{i,n})$ and $v(z_{i,n})$, $i = 1,2$ when $n \to +\infty$. We recall that $u(z)$ and $v(z)$ are defined by (3.76) and (3.77) respectively. It is clear that $u_i(x_0)$ and $v_i(x_0)$ coincide with (3.76) and (3.77) when $(z, t(z), T(z))$ are replaced by (x_0, t_i, T_i) respectively. (3.74) thus implies that

$$(1 - u_i(x_0))^2 - x_0^2 v_i(x_0)^2 \geq 0 \tag{3.97}$$

for $i = 1,2$. Using the Schwartz inequality and (3.82) as in the uniqueness proof of the solutions of Eq. (3.64) (see Proposition 3.2), it is easily seen that

$$\begin{aligned} |(1-u_0(x_0,x_0))^2 - x_0^2(v_0(x_0,x_0))^2| &\geq (1 - \sqrt{u_1(x_0)}\sqrt{u_2(x_0)})^2 - x_0^2 v_1(x_0) v_2(x_0) \\ &\geq (1-u_1(x_0))(1-u_2(x_0)) - x_0^2 v_1(x_0) v_2(x_0) \\ &\geq \sqrt{(1-u_1(x_0))^2 - x_0^2 v_1(x_0)^2}\sqrt{(1-u_2(x_0))^2 - x_0^2 v_2(x_0)^2} \geq 0. \end{aligned} \tag{3.98}$$

Therefore, $(1 - u_0(x_0,x_0))^2 - x_0^2 v_0(x_0,x_0)^2 = 0$ implies that the Schwartz inequalities and the inequalities (3.82) used to establish (3.98) are equalities. Hence, it holds that $|u_0(x_0,x_0)|^2 = u_1(x_0)u_2(x_0)$, or equivalently $|\frac{1}{M}\mathrm{Tr}(RT_1RT_2)| = (\frac{1}{M}\mathrm{Tr}(RT_1T_1^*R))^{1/2}(\frac{1}{M}\mathrm{Tr}(RT_2T_2^*R))^{1/2}$. This implies that $T_1 = aT_2^*$ for some constant $a \in \mathbb{C}$. Moreover, as $t_i = \frac{1}{M}\mathrm{Tr}(RT_i)$ for $i = 1,2$, it must hold that $t_1 = at_2^*$. (3.98) follows from (3.82) $\{a = b = 1,\ c = u_1(x_0),\ d = u_2(x_0)\}$ and $\{a = (1-u_1(x_0))^2,\ b = (1-u_2(x_0))^2,\ c = x_0^2 v_1^2,\ d = x_0^2 v_2^2\}$. Since all these terms are positive real numbers, $\sqrt{ab} - \sqrt{cd} = \sqrt{a-c}\sqrt{b-d}$ if and only if $ad = bc$. It gives us

$$\begin{aligned} &u_1(x_0) = u_2(x_0), \\ &(1-u_1(x_0))^2 x_0^2 v_2(x_0)^2 = (1-u_2(x_0))^2 x_0^2 v_1(x_0)^2. \end{aligned} \tag{3.99}$$

Since $x_0 \neq 0$ and $v_1(x_0) > 0$, the inequality $(1-u_1(x_0))^2 - x_0^2 v_1(x_0)^2 \geq 0$ implies that $u_1(x_0) \neq 1$. Hence, $u_1(x_0) < 1$ and (3.99) implies that $v_1(x_0) = v_2(x_0)$. From the definition of u_i and v_i one can notice that $u_i(x_0) = c^2 x_0^2 |t_i|^2 v_i(x_0)$. Which gives us immediately $|t_1|^2 = |t_2|^2$ and, as a consequence, $|a| = 1$. Using once again the fact that $v_1(x_0) = v_2(x_0)$ and $T_1 = aT_2^*$, we obtain that

$$\frac{|a|^2 \frac{1}{M}\mathrm{Tr}(T_2^* RRT_2)}{|1 - x_0 c^2 a^2 (t_2^*)^2|^2} = \frac{\frac{1}{M}\mathrm{Tr}(RT_2T_2^*R)}{|1 - x_0 c^2 t_2^2|^2}.$$

The numerators of both sides are equal and non zero, from what follows that the denominators are also equal, i.e.

$$|1 - x_0 c^2 a^2 (t_2^*)^2| = |1 - x_0 c^2 t_2^2|.$$

We remark that if w and z satisfy $|1-w| = |1-z|$ and $|w| = |z|$, then, either $w = z$, either $w = \bar{z}$. We use this remark for $w = x_0 c^2 t_2^2$ and $z = x_0 c^2 a^2 (t_2^*)^2$. If $w = z$, it holds that $a^2(t_2^*)^2 = t_2^2 \Rightarrow t_1^2 = t_2^2$ and since $\mathrm{Im}t_i \geq 0$ we conclude $t_1 = t_2$. If $w = \bar{z}$, we have $a^2(t_2^*)^2 = (t_2^*)^2$. If $t_2 = 0$ then it also holds that $t_1 = 0$. Otherwise, we have $a = \pm 1$. If $a = 1$, the condition $\mathrm{Im}t_i \geq 0$, leads to the conclusion that t_1 and t_2 are real and coincide. We finally consider the case $a = -1$. We recall $T_1 = aT_2^* = -T_2^*$. Therefore, it holds that

$$x_0 I_M - \frac{x_0 t_2^*}{1 - x_0 c^2 (t_2^*)^2} R = -x_0 I_M - \frac{x_0 t_2^*}{1 - x_0 c^2 (t_2^*)^2} R,$$

which is impossible, since $x_0 \neq 0$. This completes the proof of Lemma (3.9). ■

Lemmas 3.9 and 3.8, and their corresponding proofs imply the following result.

Proposition 3.4. *For each $x > 0$, $\lim_{z \to x, z \in \mathbb{C}^+} t(z) = t(x)$ exists. Moreover, $1 - x(ct(x))^2 \neq 0$, and matrix $(I + \frac{ct(x)}{1-x(ct(x))^2} R)$ is invertible. Therefore, $\lim_{z \to x, z \in \mathbb{C}^+} T(z) = T(x)$ where $T(x)$ represents matrix $T(x) = \left(-x(I + \frac{ct(x)}{1-x(ct(x))^2} R)\right)^{-1}$. Moreover, $t(x)$ is solution of the equation*

$$t(x) = \frac{1}{M}\mathrm{Tr}(RT(x)). \tag{3.100}$$

If $u(x)$ and $v(x)$ represent the terms defined by (3.76) and (3.77) for $z = x$, then it holds that

$$1 - u(x) > 0 \tag{3.101}$$

and

$$(1 - u(x))^2 - x^2(v(x))^2 \geq 0 \tag{3.102}$$

for each $x \neq 0$. Moreover, the inequality (3.102) is strict if $x \in \mathbb{R}^+ - \mathcal{S}$. If moreover $\mathrm{Im}(t(x)) > 0$, then, we have

$$1 - u(x) - xv(x) = 0. \tag{3.103}$$

Proof. It just remains to justify (3.101), (3.102), and (3.103). As function $z \to t(z)$ is analytic on $\mathbb{C} - \mathcal{S}$, $x \to t(x)$ is differentiable on $\mathbb{R}^+ - \mathcal{S}$. As $(t(x))' > 0$ and $(xt(x))' > 0$ hold on $\mathbb{R}^+ - \mathcal{S}$, the arguments used in the context of Remark 3.1 are also valid on $\mathbb{R}^+ - \mathcal{S}$, thus justifying (3.101) and the strict inequality in (3.102). $1 - u(x) \geq 0$ and inequality (3.102) also hold on $\mathcal{S} - \{0\}$ by letting $z \to x$, $z \in \mathbb{C}^+$ in Proposition 3.1. As $v(x) > 0$ for each $x \neq 0$, the strict inequality (3.101) is a consequence of (3.102).

In order to prove (3.103), we use the second component of (3.78), and remark that it implies that

$$\mathrm{Im}(t(x)) = (u(x) + xv(x))\, \mathrm{Im}(t(x)).$$

Therefore, $\mathrm{Im}(t(x)) > 0$ leads to (3.103). ■

We also add the following useful result which shows that the real part of $t(x)$ is negative for each $x > 0$.

Proposition 3.5. *For each $x \in \mathbb{R}^{+*}$, it holds that $\mathrm{Re}(t(x)) < 0$.*

Proof. It is easily checked that

$$\begin{pmatrix} \mathrm{Re}(t(z) \\ \mathrm{Re}(zt(z)) \end{pmatrix} = \begin{pmatrix} u(z) & -v(z) \\ -|z|^2 v(z) & u(z) \end{pmatrix} \begin{pmatrix} \mathrm{Re}(t(z) \\ \mathrm{Re}(zt(z)) \end{pmatrix} + \begin{pmatrix} -\mathrm{Re}(z)\frac{1}{M}\mathrm{Tr}(RT(z)(T(z))^*) \\ -|z|^2 \frac{1}{M}\mathrm{Tr}(RT(z)(T(z))^* \end{pmatrix} \tag{3.104}$$

for each $z \in \mathbb{C} - \mathcal{S}$. Moreover, as all the terms coming into play in (3.104) have a finite limit when $z \to x$ when $x \neq 0$, (3.104) remains valid on $\mathbb{R}^*$. For $z = x$, the first component of (3.104) leads to

$$\mathrm{Re}(t(x))(1 - u(x) + xv(x)) = -x\frac{1}{M}\mathrm{Tr}(RT(x)T(x)^*). \tag{3.105}$$

Proposition 3.4 implies that $1 - u(x) > 0$, when $x \in \mathbb{R}^*$. Therefore, $1 - u(x) + xv(x)$ is strictly positive as well, and it holds that

$$\mathrm{Re}(t(x)) = -x\, \frac{1}{1 - u(x) + xv(x)}\, \frac{1}{M}\mathrm{Tr}(RT(x)T(x)^*). \tag{3.106}$$

Therefore, $x > 0$ implies that $\mathrm{Re}(t(x)) < 0$ as expected. ■

We now study the behaviour of $t(z)$ when $z \to 0$. We first establish that $\lim_{z\to 0, z\in\mathbb{C}^+\cup\mathbb{R}^*} |t(z)| = +\infty$, and then that $\lim_{z\to 0, z\in\mathbb{C}^+\cup\mathbb{R}^*} zt(z) = 0$ if $c \leq 1$ and is strictly negative if $c > 1$. We recall that $t(x)$ for $x > 0$ is defined by $t(x) = \lim_{z\to x, z\in\mathbb{C}^+} t(z)$. For this, we establish various lemmas.

Lemma 3.10. *It holds that $\lim_{z\to 0, z\in\mathbb{C}^+\cup\mathbb{R}^*} |t(z)| = +\infty$.*

Proof. We assume that the statement of the Lemma does not hold, i.e. that it exists a sequence of elements of $\mathbb{C}^+ \cup \mathbb{R}^*$ $(z_n)_{n\geq 1}$ such that $\lim_{n\to+\infty} z_n = 0$ and $t(z_n) \to t_0$. (3.64) and (3.100) imply that

$$z_n t(z_n) = -\frac{1}{M}\sum_{l=1}^{\overline{M}} \frac{m_l \overline{\lambda}_l}{1 + \frac{ct(z_n)\overline{\lambda}_l}{1 - z_n(ct(z_n))^2}}. \tag{3.107}$$

$1 + \frac{ct(z_n)\overline{\lambda}_l}{1 - z_n(ct(z_n))^2}$ clearly converges towards $1 + ct_0\overline{\lambda}_l$. As the left hand side of (3.107) converges towards 0, for each l, $1 + ct_0\overline{\lambda}_l$ cannot vanish. Therefore, matrix $I + ct_0 R$ is invertible, and taking the limit of (3.107) gives

$$\frac{1}{M}\mathrm{Tr} R(I + ct_0 R)^{-1} = 0.$$

As $\mathrm{Im}\frac{1}{M}\mathrm{Tr}R(I+ct_0R)^{-1}$ cannot be zero if t_0 is not real, t_0 must be real. We now use the observation that $|z_n|v(z_n) \leq 1$ for each n (see Lemma 3.7 and Proposition 3.4 if $z_n \in \mathbb{C}^+ \cup \mathbb{R}^{+*}$, and Remark 3.1 if $z_n \in \mathbb{R}^{-*}$). As $|1-z_n(ct(z_n))^2|^2 \to 1$, $|z_n|v(z_n)$ bounded implies that $|z_n|\frac{1}{M}\mathrm{Tr}(RT(z_n)RT(z_n)^*)$ is bounded. It is easy to check that

$$|z_n|\frac{1}{M}\mathrm{Tr}(RT(z_n)RT(z_n)^*) = \frac{1}{|z_n|}\frac{1}{M}\mathrm{Tr}(R(I+ct_0R)^{-1}R(I+ct_0R)^{-1}) + \mathcal{O}(1).$$

Therefore, the boundedness of $|z_n|\frac{1}{M}\mathrm{Tr}(RT(z_n)RT(z_n)^*)$ implies that $\frac{1}{M}\mathrm{Tr}(R(I+ct_0R)^{-1}R(I+ct_0R)^{-1}) = 0$ which is of course impossible. ■

Lemma 3.11. *Consider a sequence $(z_n)_{n\geq 1}$ of elements of $\mathbb{C}^+ \cup \mathbb{R}^*$ such that $\lim_{n\to+\infty} z_n = 0$. Then, the set $(z_nt(z_n))_{n\geq 1}$ is bounded.*

Proof. We assume that $(z_nt(z_n))_{n\geq 1}$ is not bounded. Therefore, one can extract from $(z_n)_{n\geq 1}$ a subsequence, still denoted $(z_n)_{n\geq 1}$, such that $\lim_{n\to+\infty} |z_nt(z_n)| = +\infty$. Then,

$$\frac{ct(z_n)}{1-z_n(ct(z_n))^2} = \frac{1}{\frac{1}{ct(z_n)} - z_nt(z_n)} \to 0.$$

Therefore,

$$-\frac{1}{M}\mathrm{Tr}R\left(I + \frac{ct(z_n)}{1-z_n(ct(z_n))^2}R\right)^{-1} \to -\frac{1}{M}\mathrm{Tr}R.$$

This is a contradiction because the above term coincides with $z_nt(z_n)$ which cannot converge towards a finite limit. ■

Lemma 3.12. *Assume that $(z_{1,n})_{n\geq 1}$ and $(z_{2,n})_{n\geq 1}$ are sequences of elements of $\mathbb{C}^+\cup\mathbb{R}^*$ such that $\lim_{n\to+\infty} z_{i,n} = 0$ and $\lim_{n\to+\infty} z_{i,n}t(z_{i,n}) = \delta_i$ for $i=1,2$. Then, $\delta_1 = \delta_2$.*

Proof. We first remark that $|t(z_{i,n})| \to +\infty$ for $i=1,2$. Equation (3.64) implies immediately that

$$zt(z) = \left(zct(z) - \frac{1}{ct(z)}\right)\frac{1}{M}\mathrm{Tr}R\left(R + \frac{1}{ct(z)} - zct(z)\right)^{-1}. \tag{3.108}$$

As $\frac{1}{ct(z_{i,n})} \to 0$, $z_{i,n}ct(z_{i,n}) - \frac{1}{ct(z_{i,n})} \to c\delta_i$ for $i=1,2$. If $\delta_i \neq 0$, Eq. (3.108) thus implies that $c\frac{1}{M}\mathrm{Tr}R\left(R + \frac{1}{ct(z_{i,n})} - z_{i,n}ct(z_{i,n})\right)^{-1}$ converges towards 1, which implies that matrix $R - c\delta_iI$ is invertible. Therefore, either $\delta_i = 0$, either δ_i is a solution of the equation

$$1 = c\frac{1}{M}\mathrm{Tr}R(R-c\delta_iI)^{-1} \tag{3.109}$$

or equivalently, δ_i verifies

$$\delta_i = c\delta_i\frac{1}{M}\mathrm{Tr}R(R-c\delta_iI)^{-1}. \tag{3.110}$$

We note that the solutions of this equation are real, so that $\delta_i \in \mathbb{R}$ for $i=1,2$. Eq. (3.94) leads to

$$\begin{aligned}z_{1,n}t(z_{1,n}) - z_{2,n}t(z_{2,n}) = {} & z_{1,n}z_{2,n}v_0(z_{1,n},z_{2,n})(t(z_{1,n}) - t(z_{2,n})) \\ & + u_0(z_{1,n},z_{2,n})(z_{1,n}t(z_{1,n}) - z_{2,n}t(z_{2,n})).\end{aligned}$$

It is straightforward to check that $z_{1,n}z_{2,n}v_0(z_{1,n},z_{2,n})(t(z_{1,n}) - t(z_{2,n})) \to 0$ and that $u_0(z_{1,n},z_{2,n}) \to u_0(0,0) = c\frac{1}{M}\mathrm{Tr}R(R-c\delta_1I)^{-1}R(R-c\delta_2I)^{-1}$. Therefore, we obtain that

$$\delta_1 - \delta_2 = u_0(0,0)(\delta_1 - \delta_2). \tag{3.111}$$

We recall that $|u_0(z_{1,n},z_{2,n})| \leq \sqrt{u(z_{1,n})}\sqrt{u(z_{2,n})} \leq 1$. Moreover, we observe that $u(z_{i,n}) \to u_i(0) = c\frac{1}{M}\mathrm{Tr}R(R-c\delta_iI)^{-1}R(R-c\delta_iI)^{-1}$ and that $0 < u_i(0) \leq 1$. The Schwartz inequality leads to

$$|u_0(0,0)| \leq \sqrt{u_1(0)}\sqrt{u_2(0)} \leq 1. \tag{3.112}$$

If the Schwartz inequality (3.112) is strict, $|u_0(0,0)| < 1$, and $\delta_1 = \delta_2$. We now assume that $u_0(0,0) = \sqrt{u_1(0)}\sqrt{u_2(0)} = 1$. This implies that

$$R - c\delta_1 I = \kappa(R - c\delta_2 I)$$

for some real constant κ, or equivalently, $\overline{\lambda}_l - c\delta_1 = \kappa(\overline{\lambda}_l - c\delta_2)$ for each $l = 1, \ldots, \overline{M}$. If R is not a multiple of I, κ must be equal to 1, since otherwise, we would have $\overline{\lambda}_l = \overline{\lambda}_{l'}$ for each l, l'. $\kappa = 1$ implies immediately that $\delta_1 = \delta_2$. We finally consider the case where $R = \sigma^2 I$. Then, (3.110) implies that δ_i is solution of $\delta_i \frac{\sigma^2 c}{\sigma^2 - c\delta_i} = \delta_i$, i.e. $\delta_i = 0$ or

$$\delta_i = \sigma^2 \left(\frac{1}{c} - 1\right). \tag{3.113}$$

We now check that $\delta_1 = 0, \delta_2 = \sigma^2\left(\frac{1}{c} - 1\right)$ or $\delta_2 = 0, \delta_1 = \sigma^2\left(\frac{1}{c} - 1\right)$ is impossible. If this holds, $u_1(0)$ and $u_2(0)$ cannot be both equal to 1, and $|u_0(0,0)| < 1$. Therefore, (3.111) leads to a contradiction, and $\delta_1 = \delta_2$ is equal either to 0, either to $\sigma^2\left(\frac{1}{c} - 1\right)$. ■

Lemmas 3.11 and 3.12 imply the following corollary.

Corollary 3.4. *If $c \leq 1$, it holds that*

$$\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z t(z) = 0 \tag{3.114}$$

and that

$$\mu(\{0\}) = 0. \tag{3.115}$$

Proof. Lemmas 3.11 and 3.12 lead to the conclusion that $\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z t(z) = \delta$ where δ is either equal to 0, either coincides with a solution of the equation (3.110). In order to precise this, we remark that $t(x) > 0$ if $x < 0$ implies that $\delta \leq 0$. Therefore, δ coincides with a non positive solution of equation (3.110). If $c \leq 1$, it is clear that (3.110) has no strictly negative solutions. Therefore, (3.114) is established. (3.115) is a direct consequence of the identity

$$\mu(\{0\}) = \lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} -z t(z).$$

■

In order to address the case where $c > 1$ and to precise the behaviour of $\mathrm{Im}(t(z))$ when $z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*$ if $c \leq 1$, we have to evaluate $z(t(z))^2$ when $z \to 0$. The following Lemma holds.

Lemma 3.13.
- *If $c = 1$, it holds that* $\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} |z(t(z))^2| = +\infty$.
- *If $c < 1$,*

$$\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z(t(z))^2 = -\frac{1}{c(1-c)}. \tag{3.116}$$

- *If $c > 1$, the assumption* $\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z t(z) = \delta = 0$ *implies that* $\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z(t(z))^2 = -\frac{1}{c(1-c)}$, *a contradiction because the above limit is necessarily negative. Hence, δ is non zero and coincides with the strictly negative solution of Eq. (3.110), and $\mu(\{0\}) = -\delta$.*

Proof. (3.64) implies that

$$z(t(z))^2 = -\frac{1}{M}\mathrm{Tr} R \left(\frac{I}{t(z)} + \frac{c}{1 - z(ct(z))^2} R\right)^{-1}. \tag{3.117}$$

We assume in the course of this proof that $\delta = 0$ (if $c \leq 1$, this property holds). We first establish the first item of Lemma 3.13. We assume that $c = 1$ and that there exists a sequence $(z_n)_{n \in \mathbb{C}^+ \cup \mathbb{R}^*}$ such that $z_n \to 0$ and $z_n t(z_n)^2 \to \alpha$. As $|t(z_n)| \to +\infty$, (3.117) leads to $\alpha = \alpha - 1$, a contradiction. Therefore, if $c = 1$, $\lim_{z \to 0, \to \mathbb{C}^+ \cup \mathbb{R}^*} |z t(z)^2| = +\infty$ as expected.
We now establish the 2 last items. For this, we establish that if $c \neq 1$, then, $|z t(z)^2|$ is bounded when $z \in \mathbb{C}^+ \cup \mathbb{R}^*$ and z is close from 0. For this, we assume the existence of a sequence $(z_n)_{n \geq 1}$ of elements of $\mathbb{C}^+ \cup \mathbb{R}^*$ such that $z_n \to 0$ and $|z_n t(z_n)^2| \to +\infty$. Then, it holds that

$$1 = -\frac{1}{M}\mathrm{Tr} R \left(z_n t(z_n) I + \frac{c z_n t(z_n)^2}{1 - z_n (ct(z_n))^2} R\right)^{-1}.$$

As $|z_n t(z_n)^2| \to +\infty$, $\frac{c z_n t(z_n)^2}{1 - z_n (c t(z_n))^2} \to -\frac{1}{c}$. Condition $z_n t(z_n) \to 0$ thus implies that $c = 1$, a contradiction. Using again (3.117), we obtain immediately that if $z_n (t(z_n))^2 \to \alpha$, then $\alpha = -\frac{1}{c(c-1)}$. As $|zt(z)^2|$ remains bounded when $z \in \mathbb{C}^+ \cup \mathbb{R}^*$ is close from 0, this implies that $\lim_{z \to 0, z \in \mathbb{C}^+ \cup \mathbb{R}^*} z(t(z))^2 = -\frac{1}{c(1-c)}$ as expected. Taking $z \in \mathbb{R}^{-*}$ leads to the conclusion that the above limit is negative. When $c > 1$, this is a contradiction because $-\frac{1}{c(1-c)}$ is positive. Therefore, if $c > 1$, δ, the limit of $zt(z)$, cannot be equal to 0. Hence, δ coincides with the strictly negative solution of (3.110) and $\mu(\{0\}) = -\delta > 0$. This completes the proof of the Lemma. ■

Putting all the pieces together, we obtain the following characterization of μ_N.

Theorem 3.2. *The density $f_N(x)$ of μ_N w.r.t. the Lebesgue measure is a continuous function on $\mathbb{R}^{+*}$, and is given by $f_N(x) = \frac{1}{\pi}\mathrm{Im}(t_N(x))$ for each $x > 0$. If $c_N \leq 1$, μ_N is absolutely continuous, and if $c_N > 1$, then $d\mu_N(x) = f_N(x)dx + \mu_N(\{0\})\delta_0$. 0 belongs to $\mathcal{S}_N$, and the interior $\mathcal{S}_N^{\circ}$ of $\mathcal{S}_N$ is given by*

$$\mathcal{S}_N^{\circ} = \{x \in \mathbb{R}^+, \mathrm{Im}(t(x)) > 0\}. \tag{3.118}$$

If moreover $c_N < 1$, it holds that

$$f_N(x) \simeq \frac{1}{\pi} \frac{1}{\sqrt{x\, c_N (1 - c_N)}} \tag{3.119}$$

when $x \to 0^+$, while if $c_N = 1$,

$$f_N(x) \simeq \frac{1}{\pi} \frac{\sqrt{3}}{2} \left(\frac{1}{M} \mathrm{Tr}\, R^{-1} \right)^{-1/3} \frac{1}{x^{2/3}}. \tag{3.120}$$

Proof. $t(z)$ is not analytic in a neighbourhood of 0, hence, $0 \in \mathcal{S}$. As $\lim_{z \to x, z \in \mathbb{C}^+} t(z) = t(x)$ exists for $x \neq 0$, Theorem 2.1 of [43] implies that if $\mathcal{A} \subset \mathbb{R}^{+*}$ is a Borel set of zero Lebesgue measure, then $\mu(\mathcal{A}) = \int_{\mathcal{A}} f(x)dx = 0$. The continuity of f on $\mathbb{R}^{+*}$ is a also a consequence of [43].

We now prove (3.119). For this, we remark that (3.116) implies that

$$\lim_{x \to 0, x > 0} x(t(x))^2 = -\frac{1}{c(1-c)}. \tag{3.121}$$

As $\mathrm{Im}(t(x)) \geq 0$ for each $x \neq 0$, (3.121) implies that $t(x) \simeq \frac{i}{\sqrt{x}\sqrt{c(1-c}}$ when $x \to 0^+$, or equivalently that $\frac{1}{\pi}\mathrm{Im}(t(x)) \simeq \frac{1}{\pi} \frac{1}{\sqrt{x\, c(1-c)}}$.

It remains to establish (3.120). For this, we first prove that

$$\lim_{x \to 0, x > 0} x^2 (t(x))^3 = \left(\frac{1}{M} \mathrm{Tr}\, R_N^{-1} \right)^{-1}. \tag{3.122}$$

For this, we write (3.100) as

$$\frac{1}{M} \mathrm{Tr} R \left(-xt(x)I + \frac{1}{1 - \frac{1}{x(t(x))^2}} R \right)^{-1} = 1. \tag{3.123}$$

As $c = 1$, $xt(x) \to 0$ and $|x(t(x))^2| \to +\infty$ when $x \to 0, x > 0$. The left hand side of (3.123) can be expanded as

$$\frac{1}{M} \mathrm{Tr} R \left(-xt(x)I + \frac{1}{1 - \frac{1}{x(t(x))^2}} R \right)^{-1} = 1 - \frac{1}{x(t(x))^2} + \frac{1}{M} \mathrm{Tr}\, R^{-1}\, xt(x) + xt(x)\epsilon_1(x) + \frac{1}{x(t(x))^2} \epsilon_2(x),$$

where $\epsilon_1(x)$ and $\epsilon_2(x)$ converge towards 0 when $x \to 0, x > 0$. Therefore, (3.123) implies that

$$\frac{1}{M}\mathrm{Tr}\, R^{-1}\, xt(x) - \frac{1}{x(t(x))^2} = xt(x)\tilde{\epsilon}_1(x) + \frac{1}{x(t(x))^2}\tilde{\epsilon}_2(x),$$

where $\tilde{\epsilon}_1(x)$ and $\tilde{\epsilon}_2(x)$ converge towards 0 when $x \to 0, x > 0$. This leads immediately to (3.122). As function $x \to x^2(t(x))^3$ is continuous on $\mathbb{R}^{+*}$, it holds that

$$\lim_{x\to 0, x>0} x^{2/3} t(x) = e^{2ik\pi/3}\left(\frac{1}{M}\mathrm{Tr}\, R^{-1}\right)^{-1/3},$$

where k is equal to $0, 1$ or 2. If $k = 0$, the real part of $t(x)$ must be positive if x is close enough from 0. Lemma 3.5 thus leads to a contradiction. If $k = 2$, $\mathrm{Im}(t(x)) < 0$ for x small enough, a contradiction as well. Hence, k is equal to 1. Therefore,

$$\lim_{x\to 0, x>0} x^{2/3}\mathrm{Im}(t(x)) = \sin 2\pi/3 \left(\frac{1}{M}\mathrm{Tr}\, R^{-1}\right)^{-1/3}. \tag{3.124}$$

This completes the proof of (3.120). ■

We now show that function $x \to t(x)$ and $x \to f(x)$ possess a power series expansion in a neighbourhood of each point of $\mathcal{S}_N^{\circ}$. More precisely :

Proposition 3.6. *If $x_0 > 0$ and $\mathrm{Im}(t(x_0)) > 0$, then, t and f can be expanded as*

$$t(x) = \sum_{k=0}^{+\infty} a_k (x - x_0)^k, f(x) = \sum_{k=0}^{+\infty} b_k (x - x_0)^k$$

when $|x - x_0|$ is small enough.

Proof. As in [43] and [14], the proof is based on the holomorphic implicit function theorem (see [9]). We denote $t(x_0)$ by t_0. Then, Eq. (3.100) at point x_0 can be written as $h(x_0, t_0) = 0$ where function $h(z, t)$ is defined by

$$h(z,t) = t - \frac{1}{M}\mathrm{Tr}\left(R\left(-z(I + \frac{ct}{1 - z(ct)^2}R)\right)^{-1}\right).$$

As $x_0 > 0$ and $\mathrm{Im}(t_0) > 0$, function $(z, t) \to h(z, t)$ is holomorphic in a neighbourhood of (x_0, t_0). It is easy to check that

$$\left(\frac{\partial h}{\partial t}\right)_{x_0, t_0} = 1 - u_0(x_0, x_0) - x_0^2 v_0(x_0, x_0), \tag{3.125}$$

where we recall that functions u_0 and v_0 are given by (3.95) and (3.96). Following the proof of Lemma 3.9, we obtain immediately that $1 - u_0(x_0, x_0) - x_0^2 v_0(x_0, x_0) = 0$ implies that $T(x_0) = aT(x_0)^*$, and that $t_0 = a t_0^*$ for some $a \in \mathbb{C}$. The arguments of the above proof then lead to the conclusion that $t_0 = t_0^*$, a contradiction because $\mathrm{Im}(t(x_0)) > 0$. Hence, $\left(\frac{\partial h}{\partial t}\right)_{x_0, t_0} \neq 0$. The holomorphic implicit function theorem thus implies that it exists a function $z \to \tilde{t}(z)$, holomorphic in a neigbourhood N of x_0, verifying $\tilde{t}(x_0) = t_0$ and $h(z, \tilde{t}(z)) = 0$ for each $z \in N$. Moreover, condition $\mathrm{Im}(t_0) = \mathrm{Im}(\tilde{t}(x_0)) > 0$ implies that $\mathrm{Im}(\tilde{t}(z)) > 0$ and $\mathrm{Im}(z\tilde{t}(z)) > 0$ if $|z - x_0| < \epsilon$ for ϵ small enough. Therefore, if $z \in \mathbb{C}^+$ and $|z - x_0| < \epsilon$, it must hold that $\tilde{t}(z) = t(z)$ (see Proposition 3.2). Hence, $t(x) = \lim_{z\to x, z\in\mathbb{C}^+} t(z)$ must coincide with $\tilde{t}(x)$ when $|x - x_0| < \epsilon$. As $\tilde{t}(z)$ is holomorphic in a neighbourhood of x_0, function $x \to t(x)$ can be expanded as

$$t(x) = \sum_{k=0}^{+\infty} a_k (x - x_0)^k$$

when $|x - x_0| < \epsilon$. This immediately implies that f possesses a power series expansion in the interval $(x_0 - \epsilon, x_0 + \epsilon)$. ■

We finally use the above results in order the study measure ν_N associated to the Stieltjes transform

$$t_{N,\nu}(z) = \frac{1}{M}\mathrm{Tr} T_N(z).$$

As ν_N and μ_N are absolutely continuous one with respect each other, $d\nu_N(x)$ can also be written as $d\nu_N(x) = g_N(x)dx + \nu_N(\{0\})\delta_0$. Using the identity

$$\frac{1}{M}\mathrm{Tr}\left[-z\left(I + \frac{ct(z)}{1-z(ct(z))^2}R\right)T(z)\right] = 1.$$

we obtain immediately that

$$t_\nu(z) = -\frac{1}{z} - \frac{c(t(z))^2}{1-z(ct(z))^2}. \tag{3.126}$$

If $x > 0$, $t_\nu(x) = \lim_{z\to x, z\in\mathbb{C}^+} t_\nu(z)$ exists, and is given by the righthandside of (3.126) when $z = x$. Hence, for $x > 0$, $g(x) = \frac{1}{\pi}\mathrm{Im}(t_\nu(x))$, i.e.

$$g(x) = -\frac{1}{\pi}\frac{c\,\mathrm{Im}((t(x))^2)}{|1-x(ct(x))^2|^2}. \tag{3.127}$$

If $c > 1$, $|zt(z)^2| \to +\infty$ if $z \to 0$. (3.126) thus implies that $\nu_N(\{0\}) = \lim_{z\to 0} -zt_\nu(z)$ coincides with $1 - \frac{1}{c}$, which, of course, is not surprising. We now evaluate the behaviour of g when $x \to 0, x > 0$ and $c \leq 1$.

Proposition 3.7. *If $c < 1$, it holds that*

$$g(x) \simeq_{x\to 0} \frac{1}{\pi}\frac{1}{\sqrt{c\,(1-c)}}\frac{1}{M}\mathrm{Tr}(R^{-1})\frac{1}{\sqrt{x}} \tag{3.128}$$

while if $c = 1$, it holds that

$$g(x) \simeq_{x\to 0} \frac{1}{\pi}\frac{\sqrt{3}}{2}\left(\frac{1}{M}\mathrm{Tr}(R^{-1})\right)^{2/3}\frac{1}{x^{2/3}}. \tag{3.129}$$

Proof. Using Eq. (3.117), we obtain after some algebra that

$$z(t(z))^2 + \frac{1}{c(1-c)} \simeq_{z\to 0} \frac{1}{M}\mathrm{Tr}R^{-1}\frac{1}{c^2(1-c)^3}\frac{1}{t(z)}.$$

As $t(x) \simeq_{x\to 0, x>0} \frac{i}{\sqrt{x}\sqrt{c(1-c)}}$, we get that

$$\mathrm{Im}((t(x))^2) \simeq -i\,\frac{1}{M}\mathrm{Tr}R^{-1}\frac{1}{1-c}\frac{1}{(c(1-c))^{3/2}}\frac{1}{\sqrt{x}}.$$

Therefore, (3.127) immediately leads to (3.128). (3.129) is an immediate consequence of (3.124). ■

Proposition 3.7 means in practice that if $c_N \leq 1$, a number of eigenvalues of matrix $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ are close from 0. Moreover, the rate of convergence of g_N towards $+\infty$ is higher if $c_N = 1$, showing that in this case, the proportion of eigenvalues close to 0 is even larger than if $c_N < 1$.

We finally mention that $t_\nu(x)$ and $g(x)$ possess a power expansion around eachpoint $x_0 \in \mathcal{S}^\circ$. This is an obvious consequence of Proposition 3.6 and of the above expressions of $t_\nu(x)$ and of $g(x)$ in terms of $t(x)$.

3.6.2 Characterization of $\mathcal{S}_N$.

We denote by $w_N(z)$ the function defined by

$$w_N(z) = -\frac{(1-z(c_N t_N(z))^2)}{c_N t_N(z)} = zc_N t_N(z) - \frac{1}{c_N t_N(z)}. \tag{3.130}$$

It is clear that w is analytic on $\mathbb{C} - \mathcal{S}$, that $\mathrm{Im}(w(z)) > 0$ if $z \in \mathbb{C}^+$, that $w(x) = \lim_{z\to x, z\in\mathbb{C}^+} w(z)$ exists for each $x \in \mathbb{R}^*$, and that the limit still exists if $x = 0$. If we denote this limit by $w(0)$, then, it holds that

$w(0) = 0$ if $c \leq 1$ and that $w(0) = c\delta$ if $c > 1$, where we recall that δ is defined as the solution of (3.109). Moreover, $w(x)$ is real if and only if $t(x)$ is real. Therefore, the interior $\mathcal{S}^o$ of $\mathcal{S}$ is also given by

$$\mathcal{S}^o = \{x \in \mathbb{R}^+, \text{Im}(w(x)) > 0\}. \tag{3.131}$$

Moreover, as $t(x)'$ and $(xt(x))'$ are strictly positive if $x \in \mathbb{R} - \mathcal{S}$, the derivative $w'(x)$ of $w(x)$ w.r.t. x is also strictly positive on $\mathbb{R} - \mathcal{S}$. Using the equation $t(z) = \frac{1}{M}\text{Tr}RT(z)$, we obtain immediately that $t(z)$ can be expressed in terms of $w(z)$ as

$$t(z) = \frac{1}{z}\, w(z)\, \frac{1}{M}\text{Tr}R\,(R - w(z)I)^{-1}. \tag{3.132}$$

(3.130) implies that

$$1 + ct(z)w(z) - z(ct(z))^2 = 0. \tag{3.133}$$

Plugging (3.132) into (3.133), we obtain immediately that $w_N(z)$ verifies the equation

$$\phi_N(w_N(z)) = z, \tag{3.134}$$

where $\phi_N(w)$ is defined by

$$\phi_N(w) = c_N w^2 \frac{1}{M}\text{Tr}R_N\,(R_N - wI)^{-1}\left(c_N \frac{1}{M}\text{Tr}R_N\,(R_N - wI)^{-1} - 1\right). \tag{3.135}$$

Observe that (3.134) holds not only on $\mathbb{C} - \mathcal{S}$, but also for each $x \in \mathcal{S}$. Therefore, it holds that $\phi(w(x)) = x$ for each $x \in \mathbb{R}$. For each $x \in \mathbb{R} - \mathcal{S}$, it thus holds that $\phi^{'}(w(x))\, w^{'}(x) = 1$. Therefore, as $w^{'}(x) > 0$ if $x \in \mathbb{R} - \mathcal{S}$, $w(x)$ satisfies $\phi^{'}(w(x)) > 0$ for each $x \in \mathbb{R} - \mathcal{S}$. This implies that if $x \in \mathbb{R} - \mathcal{S}$, then $w(x)$ is a real solution of the polynomial equation $\phi(w) = x$ for which $\phi^{'}(w) > 0$. Moreover, Proposition 3.5 implies that if $x \in \mathbb{R}^+ - \mathcal{S}$, then, $t(x) = \text{Re}(t(x))$ is strictly negative. Eq. (3.132) for $z = x$ thus leads to the conclusion that if $x > 0$ does not belong to $\mathcal{S}$, then $w(x)$ also verifies $w(x)\frac{1}{M}\text{Tr}R\,(R - w(x)I)^{-1} < 0$. If $x < 0$, then, $t(x)$ is this time strictly positive and $w(x)$ still verifies $w(x)\frac{1}{M}\text{Tr}R\,(R - w(x)I)^{-1} < 0$. This discussion leads to the following Proposition.

Proposition 3.8. *If $x \in \mathbb{R} - \mathcal{S}$, then $w(x)$ verifies the following properties :*

$$\phi(w(x)) = x,\ \phi^{'}(w(x)) > 0,\ w(x)\,\frac{1}{M}\text{Tr}R\,(R - w(x)I)^{-1} < 0. \tag{3.136}$$

As shown below, if $x \in \mathbb{R} - \mathcal{S}$, the properties (3.136) characterize $w(x)$ among the set of all solutions of the equation $\phi(w) = x$ and allow to identify the support as the subset of $\mathbb{R}^+$ for which the equation $\phi(w) = x$ has no real solution satisfying the conditions (3.136). These results follow directly from an elementary study of function $w \to \phi(w)$.

We first consider the case $c \leq 1$, and identify the values of $x > 0$ for which the equation $\phi(w(x)) = x$ has a real solution verifying (3.136), and those for which such a solution does not exist. It is easily seen that if $x > 0$, all the real solutions of the equation $\phi(w) = x$ are strictly positive. Therefore, the third condition in (3.136) is equivalent to $\frac{1}{M}\text{Tr}R\,(R - w(x)I)^{-1} < 0$. We denote $\omega_{1,N} < \omega_{2,N} < \ldots < \omega_{\overline{M},N}$ the (necessarily real) $\overline{M}$ roots of $\frac{1}{M}\text{Tr}R_N(R_N - wI)^{-1} = \frac{1}{c_N}$ and by $\mu_{1,N} < \mu_{2,N} < \ldots < \mu_{\overline{M}-1,N}$ the roots of $\frac{1}{M}\text{Tr}R_N(R_N - wI)^{-1} = 0$. As $c \leq 1$, it is easily seen that $\omega_1 \geq 0$, and that $\omega_1 < \overline{\lambda}_{\overline{M}} < \mu_1 < \omega_2 < \overline{\lambda}_{\overline{M}-1} < \ldots < \mu_{\overline{M}-1} < \omega_{\overline{M}} < \overline{\lambda}_1$. It is clear that $\frac{1}{M}\text{Tr}R(R - wI)^{-1} < 0$ if and only if $w \in (\overline{\lambda}_{\overline{M}}, \mu_1) \cup \ldots \cup (\overline{\lambda}_2, \mu_{\overline{M}-1}) \cup (\overline{\lambda}_1, +\infty)$.

For $x > 0$, the equation $\phi(w) = x$ is easily seen to be a polynomial equation of degree $2\overline{M} + 1$. Therefore, $\phi(w) = x$ has $2\overline{M} + 1$ solutions. For each $x > 0$, this equation has at least $2\overline{M} - 1$ real solutions that cannot coincide with $w(x)$ if $x \in (\mathcal{S}^\circ)^c$:

- $\overline{M}$ solutions belong to $]\omega_1, \overline{\lambda}_{\overline{M}}[, \ldots,]\omega_{\overline{M}}, \overline{\lambda}_1[$. None of these solutions may correspond to $w(x)$ if $x \in (\mathcal{S}^\circ)^c$ because $\frac{1}{M}\text{Tr}R(R - wI)^{-1} > 0$ at these points.
- On each interval $]\overline{\lambda}_{\overline{M}}, \mu_1[, \ldots,]\overline{\lambda}_2, \mu_{\overline{M}-1}[$, the equation $\phi(w) = x$ has a real solution at which ϕ' is negative. Therefore, $\phi(w) = x$ has $\overline{M} - 1$ extra real solutions that are not equal to $w(x)$ if $x \in (\mathcal{S}^\circ)^c$.

As $\phi_N(w) \to +\infty$ if $w \to \overline{\lambda}_{1,N}, w > \overline{\lambda}_{1,N}$ and that $\phi_N(w) \to +\infty$ if $w \to +\infty$, it exists at least one point in $]\overline{\lambda}_{1,N}, +\infty[$ at which $\phi_N^{'}$ vanishes. This point is moreover unique because otherwise, $\phi_N(w) = x$ would have more than $2\overline{M}+1$ solutions for certain values of x. We denote by $w_{+,N}$ this point, and remark that if $x > x_{+,N} = \phi_N(w_{+,N})$, $\phi_N(w) = x$ has $2\overline{M}+1$ real solutions : the $2\overline{M}-1$ solutions that were introduced below, and 2 extra solutions that belong to $]\overline{\lambda}_1, w_+[$ and $]w_+, +\infty[$ respectively. Therefore, $w(x)$ is real, and it is easily seen that $w(x)$ coincides with the solution that belongs to $]w_+, +\infty[$. This implies that $]x_+, +\infty[\subset \mathbb{R} - \mathcal{S}$.

If $\phi^{'}(w)$ does not vanish on $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$, for each $x \in]0, x_+[$, ϕ is decreasing on these intervals. Therefore, none of the real solutions of $\phi(w) = x$ match with the properties of $w(x)$ when $x \in \mathbb{R}^+ - \mathcal{S}$. Therefore, $w(x)$ must be a complex number : $\phi(w) = x$ has thus $2\overline{M}-1$ real solutions, and a pair of complex conjugate roots : $w(x)$ is the positive imaginary part solution. In this case, $x \in \mathcal{S}^{\circ}$, and the support $\mathcal{S}$ coincides with $[0, x_+]$.

We illustrate such a behaviour when $\overline{M} = 3$. In the context of Fig. 3.1, the support is reduced to the single interval $[0, x_+]$ because $\phi^{'}(w) \neq 0$ for $w \in [\overline{\lambda}_3, \mu_1] \cup [\overline{\lambda}_2, \mu_2]$.

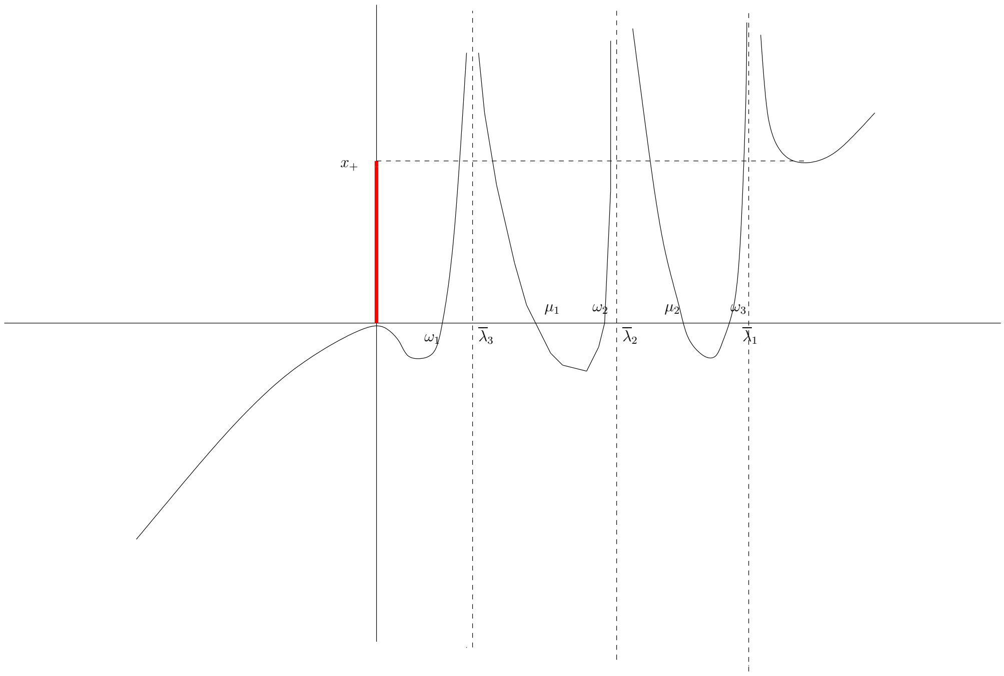

FIGURE 3.1 – Typical representation of $\phi(w)$ as a function of w for $\overline{M} = 3$. There is no local maximum on $[\overline{\lambda}_3, \mu_1]$ and on $[\overline{\lambda}_2, \mu_2]$, so that $\mathcal{S} = [0, x_+]$.

In order to precise the support when $\phi^{'}$ vanishes in $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$, we need to characterize the corresponding zeros. For this, we first justify that $\phi^{'}$ cannot have a multiplicity 2 zero. Assume for example that $\phi^{'}$ has a multiplicity 2 zero in $]\overline{\lambda}_{\overline{M}+1-l}, \mu_l[$, and denote by w_l this zero. Then, if $x_l = \phi(w_l)$, the equation $\phi(w) = x_l$ has $2\overline{M}-1$ simple real roots, and the multiplicity 3 root w_l. Therefore, the equation $\phi(w) = x_l$ has $2\overline{M}+2$ roots (counting multiplicities), a contradiction. We now establish the following useful result.

Proposition 3.9. *The number of local extrema of ϕ_N in $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$ is an even number, say $2q$, with $0 \leq q \leq \overline{M}-1$. If $q \geq 1$, we denote the arguments of these extrema by $w_{1,N}^+ < w_{2,N}^- < w_{2,N}^+ < \ldots < w_{q-1,N}^+ < w_{q,N}^-$, then $x_{1,N}^+ = \phi_N(w_{1,N}^+), x_{2,N}^- = \phi_N(w_{2,N}^-), \ldots, x_{q-1,N}^+ = \phi_N(w_{q-1,N}^+), x_{q,N}^- = \phi_N(w_{q,N}^-)$ verify*

$$x_{1,N}^+ < x_{2,N}^- < x_{2,N}^+ < \ldots < x_{q-1,N}^+ < x_{q,N}^-. \tag{3.137}$$

Moreover, for each l, the interval $]\overline{\lambda}_{\overline{M}-(l-1)}, \mu_l[$ contains at most one interval $[w_{p,N}^+, w_{p+1,N}^-]$, and $x_{p,N}^+$ (resp. $x_{p+1,N}^-$) is a local minimum (resp. local maximum) of ϕ_N.

Proof. We establish that if $w_1,\, w_2 \in \{w_1^+,\, w_2^-,\, \dots,\, w_{q-1}^+,\, w_q^-\}$ such that $w_1 > w_2$, the images $x_1 = \phi(w_1)$ and $x_2 = \phi(w_2)$ are also satisfy $x_1 > x_2$. The goal is to show that ratio $(x_1 - x_2)/(w_1 - w_2)$ is always positive. For more convenience we put $f_n = \frac{c_N}{M}\mathrm{Tr}R_N(R_N - w_n I_M)^{-1} = \frac{c_N}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{\overline{\lambda}_i - w_n}$ for $n = 1, 2$. With this and (3.135) we can rewrite

$$x_n = \phi(w_n) = w_n^2 f_n(f_n - 1) = w_n^2 p_n(p_n - 1), \tag{3.138}$$

where $p_n = 1 - f_n$. Let us notice that extremes w_1 and w_2 are by definition such that f_1 and f_2 are negative. Using directly (3.138) for x_1 and x_2 we can write

$$\begin{aligned}\frac{x_1 - x_2}{w_1 - w_2} &= \frac{(w_1^2 p_1^2 - w_2^2 p_2^2) - (w_1^2 p_1 - w_2^2 p_2)}{w_1 - w_2} \\ &= (w_1 p_1 + w_2 p_2)\frac{w_1 p_1 - w_2 p_2}{w_1 - w_2} - \frac{w_1^2 p_1 - w_2^2 p_2}{w_1 - w_2}.\end{aligned} \tag{3.139}$$

With the definition of $f_{1,2}$ the first term of (3.139) can be expanded as

$$\begin{aligned}\frac{w_1 p_1 - w_2 p_2}{w_1 - w_2} &= 1 + \frac{c}{M}\sum_{l=1}^{\bar{M}} \frac{\overline{\lambda}_i m_i}{w_1 - w_2}\left(\frac{w_2}{\overline{\lambda}_i - w_2} - \frac{w_1}{\overline{\lambda}_i - w_1}\right) \\ &= 1 - \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}.\end{aligned}$$

And similarly the second one as

$$\begin{aligned}\frac{w_1^2 p_1 - w_2^2 p_2}{w_1 - w_2} &= (w_1 + w_2) + \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{w_1 - w_2}\left(\frac{w_2^2}{\overline{\lambda}_i - w_2} - \frac{w_1^2}{\overline{\lambda}_i - w_1}\right) \\ &= (w_1 + w_2)\left(1 - \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) + w_1 w_2 \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}.\end{aligned}$$

Putting the last two equation in (3.139) we obtain

$$\begin{aligned}\frac{x_1 - x_2}{w_1 - w_2} &= (w_1 p_1 + w_2 p_2 - w_1 - w_2)\left(1 - \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) \\ &\quad - w_1 w_2 \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)} = -(w_1 f_1 + w_2 f_2) \\ &\quad \times \left(1 - \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) - w_1 w_2 \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}.\end{aligned}$$

Now we recall that $-f_n$ is positive as well as $w_1,\, w_2 > 0$ from what we have $-(w_1 f_1 + w_2 f_2) > 0$. That allows us to use the inequality

$$\frac{1}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)} \le \frac{1}{2}\left(\frac{1}{(\overline{\lambda}_i - w_1)^2} + \frac{1}{(\overline{\lambda}_i - w_2)^2}\right)$$

and to write

$$\begin{aligned}\frac{x_1 - x_2}{w_1 - w_2} &\ge -(w_1 f_1 + w_2 f_2)\left(1 - \frac{c}{2M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_1)^2} - \frac{c}{2M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w_2)^2}\right) \\ &\quad - w_1 w_2 \frac{c}{M}\sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}.\end{aligned}$$

It is easy to check that $\frac{c}{M}\sum \frac{\overline{\lambda}_i^2 m_i}{(\overline{\lambda}_i - w)^2} = f(w) + wf'(w)$. Using this we can rewrite last inequality as

$$\frac{x_1 - x_2}{w_1 - w_2} \geq -\frac{1}{2}(w_1 f_1 + w_2 f_2)\left(2 - f_1 - w_1 f_1' - f_2 - w_2 f_2'\right) \\ - w_1 w_2 \frac{c}{M} \sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}. \quad (3.140)$$

Taking the derivatives of the expression (3.138), we obtain that $\phi'(w_n) = 2w_n f_n^2 - 2w_n f_n + 2w_n^2 f_n f_n' - w_n^2 f_n'$. By definition, $w_{1,2}$ are extremes of function $\phi(w)$, i.e. $\phi'(w_{1,2}) = 0$. This gives immediately $f_n + w_n f_n' - 1 = \frac{w_n f_n'}{2f_n}$. After putting this into (3.140) and regrouping terms we obtain

$$\begin{aligned}\frac{x_1 - x_2}{w_1 - w_2} &\geq \frac{1}{4}(w_1 f_1 + w_2 f_2)\left(\frac{w_1 f_1'}{f_1} + \frac{w_2 f_2'}{f_2}\right) - w_1 w_2 \frac{c}{M} \sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)} \\ &= \frac{1}{4}(w_1^2 f_1' + w_2^2 f_2') + \frac{1}{4} w_1 w_2 \left(f_1' \frac{f_2}{f_1} + f_2' \frac{f_1}{f_2}\right) - w_1 w_2 \frac{c}{M} \sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}.\end{aligned}$$

Finally, we denote by I_1, I_2, I_3 the three terms of the r.h.s and show that $I_1 + \frac{1}{2}I_3$ and $I_2 + \frac{1}{2}I_3$ can be presented as the sum of positive terms. Using again the definition of $f_{1,2}$ we expand $I_1 + \frac{1}{2}I_3$ as

$$\begin{aligned}&\frac{1}{4}\left(w_1^2 f_1' + w_2^2 f_2' - 2w_1 w_2 \frac{c}{M} \sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) \\ &\quad = \frac{c}{4M} \sum \overline{\lambda}_i m_i \Big(\frac{w_1^2}{(\overline{\lambda}_i - w_1)^2} + \frac{w_2^2}{(\overline{\lambda}_i - w_2)^2} - \frac{2w_1 w_2}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\Big) \\ &\quad = \frac{c}{4M} \sum \overline{\lambda}_i m_i \left(\frac{w_1}{\overline{\lambda}_i - w_1} - \frac{w_2}{\overline{\lambda}_i - w_2}\right)^2.\end{aligned}$$

Similarly, $I_2 + \frac{1}{2}I_3$ can be written as

$$\begin{aligned}&\frac{1}{4} w_1 w_2 \left(f_1' \frac{f_2}{f_1} + f_2' \frac{f_1}{f_2} - 2\frac{c}{M} \sum_1^{\bar{M}} \frac{\overline{\lambda}_i m_i}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) \\ &\quad = w_1 w_2 \frac{c}{4M} \sum \overline{\lambda}_i m_i \left(\frac{f_2/f_1}{(\overline{\lambda}_i - w_1)^2} + \frac{f_1/f_2}{(\overline{\lambda}_i - w_2)^2} - \frac{2}{(\overline{\lambda}_i - w_1)(\overline{\lambda}_i - w_2)}\right) \\ &\quad = w_1 w_2 \frac{c}{4M} \sum \overline{\lambda}_i m_i \left(\frac{\sqrt{f_2/f_1}}{\overline{\lambda}_i - w_1} - \frac{\sqrt{f_1/f_2}}{\overline{\lambda}_i - w_2}\right)^2.\end{aligned}$$

This shows that $x_1 - x_2 > 0$, and that (3.137) holds. It remains to justify that each interval $(]\overline{\lambda}_{\overline{M}-(l-1)}, \mu_l[)_{l=1,\ldots,\overline{M}-1}$ contains at most one interval $[w_{p,N}^+, w_{p+1,N}^-]$. Assume that the interval $]\overline{\lambda}_{\overline{M}-(l-1)}, \mu_l[$ contains 2 intervals $[w_{p_1,N}^+, w_{p_1+1,N}^-]$ and $[w_{p_2,N}^+, w_{p_2+1,N}^-]$ with $p_1 < p_2$. Then, it also holds that $[w_{p_1+1,N}^+, w_{p_1+2,N}^-] \subset]\overline{\lambda}_{\overline{M}-(l-1)}, \mu_l[$. $x_{p_1,N}^+$ is necessarily a local minimum because $x_{p_1,N}^+ < x_{p_1+1,N}^-$ while $x_{p_1+1,N}^-$ must be a local maximum. The same property holds for $x_{p_1+1,N}^+$ and $x_{p_1+2,N}^-$. However, this contradicts the property $x_{p_1+1,N}^- < x_{p_1+1,N}^+$. This completes the proof of Proposition 3.9. ■

Proposition 3.9 allows to identify the support $\mathcal{S}_N$.

Corollary 3.5. *When $c_N \leq 1$, the support $\mathcal{S}_N$ is given by*

$$\mathcal{S}_N = [0, x_{1,N}^+] \cup [x_{2,N}^-, x_{2,N}^+] \cup \ldots [x_{q,N}^-, x_{+,N}]. \quad (3.141)$$

Proof. If x belongs to the interior of the righthandside of (3.141), $\phi(w) = x$ has only $2\overline{M} - 1$ real solutions. This implies that the 2 remaining roots are complex valued, i.e. that $x \in \mathcal{S}^\circ$. This leads to the conclusion that

$$]0, x^+_{1,N}[\cup]x^-_{2,N}, x^+_{2,N}[\cup \ldots]x^-_{q,N}, x_{+,N}[\subset \mathcal{S}^\circ$$

and that

$$[0, x^+_{1,N}] \cup [x^-_{2,N}, x^+_{2,N}] \cup \ldots [x^-_{q,N}, x_{+,N}] \subset \mathcal{S}.$$

Conversely, if $x \in \mathbb{R}^+ - \left([0, x^+_{1,N}] \cup [x^-_{2,N}, x^+_{2,N}] \cup \ldots [x^-_{q,N}, x_{+,N}]\right)$, the equation $\phi(w) = x$ has $2\overline{M} + 1$ real solutions, which implies that $w(x)$ is real. Therefore,

$$\mathbb{R}^+ - \left([0, x^+_{1,N}] \cup [x^-_{2,N}, x^+_{2,N}] \cup \ldots [x^-_{q,N}, x_{+,N}]\right) \subset \mathbb{R}^+ - \mathcal{S}$$

or equivalently,

$$\mathcal{S} \subset [0, x^+_{1,N}] \cup [x^-_{2,N}, x^+_{2,N}] \cup \ldots [x^-_{q,N}, x_{+,N}].$$

This completes the proof of Corollary (3.5). ■

We illustrate the above behaviour when $\overline{M} = 3$. In the context of Fig. 3.2, ϕ' vanishes on $[\overline{\lambda}_3, \mu_1]$ and not on $[\overline{\lambda}_2, \mu_2]$. The support thus coincides with $\mathcal{S} = [0, x_1^+] \cup [x_2^-, x_+]$.

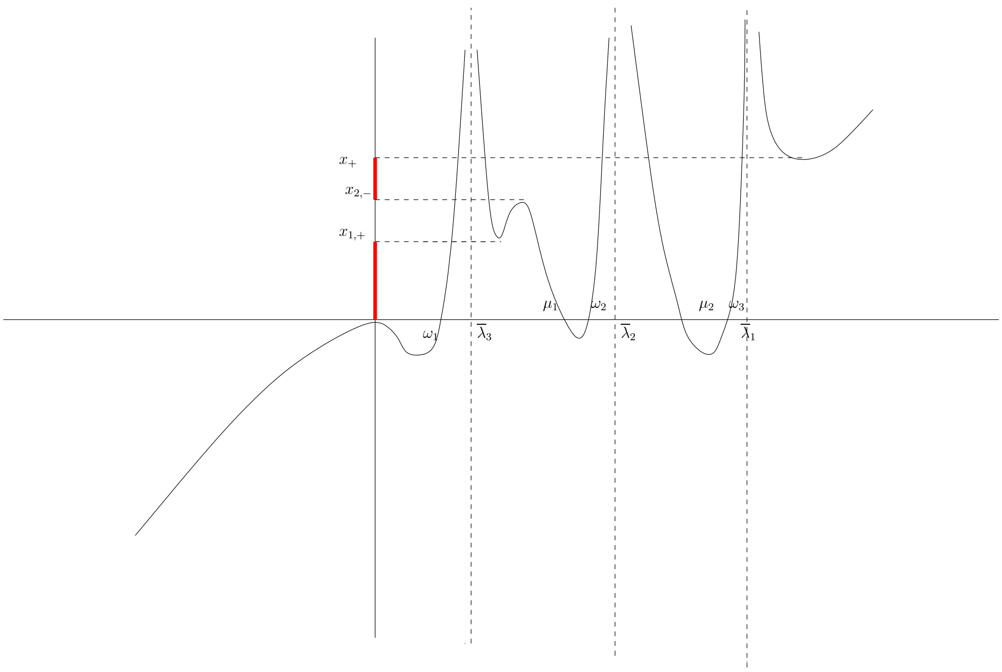

FIGURE 3.2 – Typical representation of $\phi(w)$ as a function of w for $\overline{M} = 3$. There are 2 local extrema on $[\overline{\lambda}_3, \mu_1]$ and no local maximum on $[\overline{\lambda}_2, \mu_2]$, so that $\mathcal{S} = [0, x_1^+] \cup [x_2^-, x_+]$.

When matrix R_N is reduced to $R_N = \sigma^2 I$, i.e. $\overline{M} = 1$ and $\overline{\lambda}_1 = \sigma^2$, the support of course coincides with $\mathcal{S}_N = [0, x_{+,N}]$, and $x_{+,N}$ is given by

$$x_{+,N} = \sigma^4 c_N \left(1 + \frac{1}{\frac{1+\sqrt{1+8c_N}}{2}}\right)^2 \left(c_N + \frac{1+\sqrt{1+8c_N}}{2}\right). \tag{3.142}$$

Moreover, $w_{+,N}$ is equal to

$$w_{+,N} = \sigma^2 \left(1 + \frac{1+\sqrt{1+8c_N}}{2}\right). \tag{3.143}$$

(3.142) and (3.143) are in accordance with the results of [28].

We now briefly address the case $c_N > 1$. The behaviour of ϕ_N is essentially the same as if $c_N \leq 1$, except that the first root $\omega_{1,N}$ of the equation $\frac{1}{M}\mathrm{Tr}R_N(R_N - wI)^{-1} = \frac{1}{c_N}$ is now strictly negative. As $\phi_N(0) = 0$, this implies that it exists $w_{-,N} \in (\omega_{1,N}, 0)$ for which $\phi_N^{'}(w_{-,N}) = 0$. Moreover, this point is unique, otherwise, the equation $\phi_N(w) = x$ would have more than $2\overline{M} + 1$ roots for certain values of $x > 0$. $x_{-,N} = \phi_N(w_{-,N}) > 0$ is thus a local maximum of ϕ_N whose argument is strictly negative. We also notice that $\phi_N(w) > 0$ if $0 < w < \overline{\lambda}_{\overline{M}}$. Apart these differences, the behaviour of ϕ_N for $w > \overline{\lambda}_{\overline{M}}$ remains the same as if $c_N \leq 1$. In particular, Proposition 3.9 still holds true. However, we remark that if $0 < x < x_{-,N}$, the equation $\phi_N(w) = x$ has still $2\overline{M} - 1$ real solutions that are strictly positive, and 2 extra real roots, the smallest one being less than $w_{-,N}$ and the other one being negative and largest that $w_{-,N}$. This implies that $w_N(x)$ is real. We also notice that $w_N(x)$ coincides with the smallest extra negative root because it satisfies conditions (3.136). Hence, the interval $]0, x_{-,N}[$ is included into $\mathbb{R}^+ - \mathcal{S}_N$. If $\phi_N^{'}$ does not vanish on $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$, for $x \in]x_{-,N}, x_{+,N}[$, the equation $\phi_N(w) = x$ has only $2\overline{M} - 1$ real solutions that do not satisfy conditions (3.136) and 2 extra complex conjugates solutions. Therefore, $]x_{-,N}, x_{+,N}[\subset \mathcal{S}_N^{\circ}$ and $[x_{-,N}, x_{+,N}] \subset \mathcal{S}_N$. Conversely, $]0, x_{-,N}[\cup]x_{+,N}, +\infty[\subset \mathbb{R}^+ - \mathcal{S}_N$, which implies that $\mathcal{S}_N \subset \{0\} \cup [x_{-,N}, x_{+,N}]$. As it was established above that $\{0\} \subset \mathcal{S}_N$, we deduce that $\mathcal{S}_N = \{0\} \cup [x_{-,N}, x_{+,N}]$ if $\phi_N^{'}$ does not vanish on $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$. If $\phi_N^{'}$ vanishes on $]\overline{\lambda}_{\overline{M}}, \mu_1[\cup \ldots \cup]\overline{\lambda}_2, \mu_{\overline{M}-1}[$, i.e. if $q \geq 1$ (we recall that q is defined in Proposition 3.9), the support is given by

$$\mathcal{S}_N = \{0\} \cup [x_{-,N}, x_{1,N}^+] \cup [x_{2,N}^-, x_{2,N}^+] \cup \ldots [x_{q,N}^-, x_{+,N}]. \tag{3.144}$$

To justify this, we just need to establish that $x_{-,N} < x_{1,N}^+$, and to use the same arguments as in the proof of Corollary 3.5. To justify $x_{-,N} < x_{1,N}^+$, we put $w_1 = w_{-,N}, w_2 = w_{1,N}^+$, and follow step by step the arguments used to evaluate $\phi(w_2) - \phi(w_1) > 0$. We notice that in contrast with the context of the proof of Corollary 3.5, $w_1 < 0$ and $f_1 > 0$. However, $f_1 w_1$ is still negative, so that $-(w_1 f_1 + w_2 f_2)$ is still positive. This allows to conclude that all the inequalities used in the course of the proof of Corollary 3.5 remain valid, except the evaluation of the term $I_2 + I_3/2$ that needs the following simple modification : we express $I_2 + I_3/2$ as

$$-w_1 w_2 \frac{c}{4M} \sum \lambda_i m_i \times \left(\frac{-f_2/f_1}{(\lambda_i - w_1)^2} + \frac{-f_1/f_2}{(\lambda_i - w_2)^2} + \frac{2}{(\lambda_i - w_1)(\lambda_i - w_2)} \right).$$

As $-f_2/f_1$ and $-f_1/f_2$ are positive, it holds that

$$I_2 + I_3/2 = -w_1 w_2 \frac{c}{4M} \sum \lambda_i m_i \left(\frac{\sqrt{-f_2/f_1}}{\lambda_i - w_1} + \frac{\sqrt{-f_1/f_2}}{\lambda_i - w_2} \right)^2.$$

Therefore, $I_2 + I_3/2 > 0$, and $\phi(w_2) - \phi(w_1) > 0$ holds.

In order to unify the cases $c_N \leq 1$ and $c_N > 1$, we define $x_{-,N}$ for $c_N \leq 1$ by $x_{-,N} = 0$, and summarize the above discussion by the following result.

Theorem 3.3. *The support $\mathcal{S}_N$ is given by*

$$\mathcal{S}_N = \{0\}\mathbb{I}_{c_N > 1} \cup [x_{-,N}, x_{1,N}^+] \cup [x_{2,N}^-, x_{2,N}^+] \cup \ldots [x_{q,N}^-, x_{+,N}]. \tag{3.145}$$

We now establish that sequences $(w_{+,N})_{N \geq 1}$ and $(x_{+,N})_{N \geq 1}$ are bounded. In other words, for each N, the support $\mathcal{S}_N$ is included into a compact interval that does not depend on N.

Lemma 3.14.

$$\sup_{N \geq 1} w_{+,N} < +\infty, \ \sup_{N \geq 1} x_{+,N} < +\infty. \tag{3.146}$$

Proof. In order to prove this lemma, we use that $w_{+,N} > \lambda_{1,N}$ and that $\phi_N^{'}(w_{+,N}) = 0$. It is easy to check that

$$\begin{aligned}\phi_N^{'}(w) = 2c_N^2 w \frac{1}{M}\mathrm{Tr}R(wI - R)^{-1} - (c_N w)^2 \frac{1}{M}\mathrm{Tr}R(wI - R)^{-2} \\ - 2c_N^2 w \left(\frac{1}{M}\mathrm{Tr}R(wI - R)^{-1}\right)^2 - 2(c_N w)^2 \frac{1}{M}\mathrm{Tr}R(wI - R)^{-2} \frac{1}{M}\mathrm{Tr}R(wI - R)^{-1}.\end{aligned}$$

For $w > b > \lambda_{1,N}$, it is clear that $\|(wI-R)^{-1}\| \leq \frac{1}{w-b}$. Writing that $w\frac{1}{M}\mathrm{Tr}R(wI-R)^{-1} = \frac{1}{M}\mathrm{Tr}R + \frac{1}{M}\mathrm{Tr}R^2(wI-R)^{-1}$ and $w^2\frac{1}{M}\mathrm{Tr}R(wI-R)^{-2} = \frac{1}{M}\mathrm{Tr}R + w\left(\frac{1}{M}\mathrm{Tr}R(wI-R)^{-2}\right) - \frac{1}{M}\mathrm{Tr}R^2(wI-R)^{-1}$, we obtain immediately that $\phi_N^{'}(w)$ can be written as

$$\phi_N^{'}(w) = c_N^2 \frac{1}{M}\mathrm{Tr}R + \delta_N(w),$$

where $\delta_N(w)$ verifies $|\delta_N(w)| \leq \delta(w)$ and $w \to \delta(w)$ is a rational function of w that does not depend on N and which converges towards 0 when $w \to +\infty$. Therefore, for each $\eta > 0$, it exists $w_1 > b$ such that $\phi_N^{'}(w) > c_N^2 \frac{1}{M}\mathrm{Tr}R - \eta$ for each $w \geq w_1$. As $c_N \to c_*$ and that $\frac{1}{M}\mathrm{Tr}R \geq a$, we obtain that $\phi_N^{'}(w) > \frac{c_*^2}{2}\, a$ for $w \geq w_1$. As $\phi_N^{'}(w_{+,N}) = 0$, we deduce from this that $w_{+,N} < w_1$. As w_1 does not depend on N, this establishes that $\sup_{N\geq 1} w_{+,N} < +\infty$. To prove that $x_{+,N}$ is bounded, we observe that $x_{+,N} = \phi_N(w_{+,N}) < \phi_N(w_1)$. As $w_1 > b$, it is easily seen that

$$\phi_N(w_1) < 2c_N^2 w_1^2 \left(\frac{b}{(w_1-b)^2} + \frac{b}{(w_1-b)} \right).$$

Therefore, sequences $(\phi_N(w_1))_{N\geq 1}$ and $(x_{+,N})_{N\geq 1}$ are bounded. This completes the proof of Lemma 3.14. ■

We finally provide a sufficient condition under which the support is reduced to $\mathcal{S}_N = [0, x_{+,N}]$ if $c_N < 1$ and to $\mathcal{S}_N = \{0\} \cup [x_{-,N}, x_{+,N}]$ if $c_N > 1$. More precisely, the following result holds.

Proposition 3.10. *Assume that there exists $\kappa > 0$ such that for each M large enough, the following condition holds :*

$$|\lambda_{k,N} - \lambda_{l,N}| \leq \kappa \left(\frac{|k-l|}{M} \right)^{1/2} \tag{3.147}$$

for each pair (k,l), $1 \leq k \leq l \leq M$. Then, for each M large enough, $\mathcal{S}_N = [0, x_{+,N}]$ if $c_N \leq 1$ and to $\mathcal{S}_N = \{0\} \cup [x_{-,N}, x_{+,N}]$ if $c_N > 1$.

Proof. We assume that (3.147) holds, and that $\mathcal{S}$ does not coincide with $[0, x_+]$ or $\mathcal{S} = \{0\} \cup [x_-, x_+]$, i.e. $\phi^{'}(w)$ vanishes at a point w_0 such that $\lambda_1 < w_0 < \lambda_M$ and $\frac{1}{M}\mathrm{Tr}R(R-w_0I)^{-1} < 0$. After some algebra, we obtain that w_0 satisfies :

$$\frac{1}{M}\mathrm{Tr}\left(R(R-w_0I)^{-1}\right)^2 = \frac{-\frac{1}{M}\mathrm{Tr}R(R-w_0I)^{-1}}{1-2c\frac{1}{M}\mathrm{Tr}R(R-w_0I)^{-1}}.$$

As $\frac{1}{M}\mathrm{Tr}R(R-w_0I)^{-1} < 0$, this implies that

$$\frac{1}{M}\mathrm{Tr}\left(R(R-w_0I)^{-1}\right)^2 = \frac{1}{M}\sum_{k=1}^{M}\left(\frac{\lambda_k}{\lambda_k - w_0}\right)^2 < -\frac{1}{M}\mathrm{Tr}R(R-w_0I)^{-1}$$

$$\leq \frac{1}{M}\sum_{k=1}^{M}\frac{\lambda_k}{|\lambda_k - w_0|}.$$

Jensen's inequality leads to $\left(\frac{1}{M}\sum_{k=1}^{M}\frac{\lambda_k}{|\lambda_k-w_0|}\right)^2 \leq \frac{1}{M}\sum_{k=1}^{M}\left(\frac{\lambda_k}{\lambda_k-w_0}\right)^2$. Therefore, we obtain that $\frac{1}{M}\sum_{k=1}^{M}\frac{\lambda_k}{|\lambda_k-w_0|} < 1$, and that

$$\frac{1}{M}\sum_{k=1}^{M}\left(\frac{\lambda_k}{\lambda_k - w_0}\right)^2 < 1. \tag{3.148}$$

We assume that $\lambda_{j_0} < w_0 < \lambda_{j_0+1}$. Then, hypothesis (2.2) and condition (3.147) imply that

$$\left(\frac{\lambda_k}{\lambda_k - w_0}\right)^2 > \frac{a^2}{\kappa^2}\frac{M}{(|k-j_0|+1)}.$$

Hence, it must hold that

$$\frac{a^2}{\kappa^2}\sum_{k=1}^{M}\frac{1}{(|k-j_0|+1)} < 1$$

for each M large enough, a contradiction because $\sum_{k=1}^{M}\frac{1}{(|k-j_0|+1)}$ is easily seen to be an unbounded term. ■

3.7 No eigenvalues outside the support.

In this paragraph, we establish the following result :

Theorem 3.4. *Assume that there exists $\epsilon > 0$, $\kappa_1 \in \mathbb{R}$, $\kappa_2 \in \mathbb{R} \cup \{+\infty\}$, $\kappa_2 > \kappa_1$ and an integer N_0 such that*

$$(\kappa_1 - \epsilon, \kappa_2 + \epsilon) \cap \mathcal{S}_N = \emptyset \qquad \forall N \geq N_0. \tag{3.149}$$

Then with probability one, no eigenvalues of $W_{f,N}W_{p,N}^* W_{p,N} W_{f,N}^*$ appears in $[\kappa_1, \kappa_2]$ for all N large enough.

We first remark that it is sufficient to consider the case where $\kappa_2 < +\infty$. To justify this claim, we recall that $\cup_{N \geq 1} \mathcal{S}_N$ is a compact subset (see Lemma 3.14), and notice that $\|W_{f,N}W_{p,N}^* W_{p,N} W_{f,N}^*\| \leq \|W_N\|^4$ where matrix W_N is defined by (3.4). Moreover, (3.5) implies that almost surely, for N large enough, $\|W_N\|^2 \leq b\,(1+\delta+\sqrt{c_*})^2$ where $\delta > 0$. Therefore, almost surely, the largest eigenvalue of $W_{f,N}W_{p,N}^* W_{p,N} W_{f,N}^*$ is, for each N large enough, upperbounded by the nice constant $b^2\,(1+\delta+\sqrt{c_*})^4$. This justifies that it is sufficient to assume that $\kappa_2 < +\infty$ in the following.

In order to establish Theorem 3.4, we use the Haagerup-Thornbjornsen approach ([17], see also [7]). The crucial step of the proof is the following Proposition.

Proposition 3.11. $\forall z \in \mathbb{C}^+$, *we have for N large enough,*

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}Q_N(z)\right\} = \frac{1}{M}\mathrm{Tr}T_N(z) + \frac{1}{N^2} r_N(z), \tag{3.150}$$

where r_N is holomorphic in $\mathbb{C}^+$ and satisfies

$$|r_N(z)| \leq P_1(|z|) P_2\left(\frac{1}{\mathrm{Im} z}\right) \tag{3.151}$$

for each $z \in \mathbb{C}^+$, where P_1 and P_2 are nice polynomials.

Proof. To prove (3.150) we write

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}Q_N(z)\right\} - \frac{1}{M}\mathrm{Tr}T_N(z) = \frac{1}{ML}\mathrm{Tr}\left[\mathbb{E}\left\{Q_N(z)\right\} - I_L \otimes S_N(z)\right] + \frac{1}{M}\mathrm{Tr}\left[S_N(z) - T_N(z)\right].$$

As (3.63) holds, it is sufficient to establish that

$$\left|\frac{1}{M}\mathrm{Tr}[S_N(z) - T_N(z)]\right| \leq \frac{1}{N^2} P_1(|z|) P_2(\mathrm{Im}^{-1} z) \tag{3.152}$$

for some nice polynomial P_1 and P_2. In the following, we denote by $s_N(z)$ the function defined by

$$s_N(z) = \frac{1}{M}\mathrm{Tr}R_N S_N(z). \tag{3.153}$$

It is clear that $s_N \in \mathcal{S}(\mathbb{R}^+)$. Moreover, if $\mu_{N,s}$ represents the associated positive measure, then we have

$$\mu_{N,s}(\mathbb{R}^+) = \frac{1}{M}\mathrm{Tr}R_N, \; \int_{\mathbb{R}^+} \lambda\, d\mu_{N,s}(\lambda) = c_N \frac{1}{M}\mathrm{Tr}R_N \frac{1}{M}\mathrm{Tr}R_N^2 \tag{3.154}$$

(3.154) can be proved using the arguments of the proof of Proposition 3.2.
As $\frac{1}{M}\mathrm{Tr}[S_N(z) - T_N(z)]$ is given by (3.86) for $F = I$, (3.152) appears equivalent to the property

$$\left|\frac{1}{M}\mathrm{Tr}[R_N(S_N(z) - T_N(z))]\right| = |s_N(z) - t_N(z)| \leq \frac{1}{N^2} P_1(|z|) P_2(\mathrm{Im}^{-1} z). \tag{3.155}$$

In order to prove (3.155), we define the following functions that appear formally similar to functions $u(z)$ and $v(z)$ defined by (3.76) and (3.77) :

$$
\begin{aligned}
u_\alpha(z) &= c\frac{|cz\alpha(z)|^2 \frac{1}{M}\mathrm{Tr}(RS(z)S^*(z)R)}{|1-z(c\alpha(z))^2|^2}, \\
v_\alpha(z) &= c\frac{\frac{1}{M}\mathrm{Tr}(RS(z)S^*(z)R)}{|1-z(c\alpha(z))^2|^2}, \\
u_{t,\alpha}(z) &= c\frac{|cz|^2 t(z)\alpha(z)\frac{1}{M}\mathrm{Tr}(RS(z)T(z)R)}{(1-z(c\alpha(z))^2)(1-z(ct(z))^2)}, \qquad (3.156)\\
v_{t,\alpha}(z) &= c\frac{\frac{1}{M}\mathrm{Tr}(RS(z)T(z)R)}{(1-z(c\alpha(z))^2)(1-z(ct(z))^2)}. \qquad (3.157)
\end{aligned}
$$

Using equation $t(z) = \frac{1}{M}\mathrm{Tr}RT(z)$ and the definition of $s(z)$ and $S(z)$, we obtain easily that

$$
\begin{pmatrix} (s(z)-t(z)) \\ z(s(z)-t(z)) \end{pmatrix} = \mathbf{D}_{t,\alpha}(z)\begin{pmatrix} (s(z)-t(z)) \\ z(s(z)-t(z)) \end{pmatrix} + \begin{pmatrix} \epsilon_1(z) \\ \epsilon_2(z) \end{pmatrix}
$$

holds, where

$$
\begin{aligned}
\epsilon_1(z) &= (\alpha(z)-s(z))(zv_{t,\alpha}(z)+u_{t,\alpha}(z)), \\
\epsilon_2(z) &= z(\alpha(z)-s(z))(zv_{t,\alpha}(z)+u_{t,\alpha}(z)), \\
\mathbf{D}_{t,\alpha}(z) &= \begin{pmatrix} u_{t,\alpha}(z) & v_{t,\alpha}(z) \\ z^2 v_{t,\alpha}(z) & u_{t,\alpha}(z) \end{pmatrix}.
\end{aligned}
$$

This can also be written as

$$
(\mathbf{I}-\mathbf{D}_{t,\alpha}(z))\begin{pmatrix} (s(z)-t(z)) \\ z(s(z)-t(z)) \end{pmatrix} = \begin{pmatrix} \epsilon_1(z) \\ \epsilon_2(z) \end{pmatrix}. \qquad (3.158)
$$

The application of (3.62) to $F = I_L \otimes R$ leads to $\alpha(z) - s(z) = \mathcal{O}_z(N^{-2})$. In order to verify that $(\epsilon_i(z))_{i=1,2}$ are $\mathcal{O}_z(N^{-2})$ as well, we have to control $u_{t,\alpha}$ and $v_{t,\alpha}$. As $t(z), \alpha(z), \|T(z)\|$ and $\|S(z)\|$ are $\mathcal{O}_z(1)$ terms, it is sufficient to evaluate the denominator of the right handside of (3.156). As the mass and the first moment of μ and $\overline{\mu}$ (the measure associated to $\alpha(z)$) both verify the conditions of Lemma 3.5, this Lemma implies that $(1-z(ct(z))^2)^{-1} = \mathcal{O}_z(1)$ and $(1-z(c\alpha(z))^2)^{-1} = \mathcal{O}_z(1)$. Therefore, we have checked that $(\epsilon_i(z))_{i=1,2}$ are $\mathcal{O}_z(N^{-2})$ terms.

In order to evaluate $s(z)-t(z)$, it is of course necessary to show that matrix $I - \mathbf{D}_{t,\alpha}(z)$ is invertible on $\mathbb{C}^+$, and to control the action of its inverse on the vector $(\epsilon_1(z), \epsilon_1(z))^T$. We define matrix $\mathbf{D}_\alpha$ by

$$
\mathbf{D}_\alpha(z) = \begin{pmatrix} u_\alpha(z) & v_\alpha(z) \\ z^2 v_\alpha(z) & u_\alpha(z) \end{pmatrix}
$$

and establish the following result.

Lemma 3.15. *For each $z \in \mathbb{C}^+$, it exist nice constants κ and β such that*

$$
\det\,(I-\mathbf{D}(z)) \geq \frac{\kappa\,(\mathrm{Im}z)^8}{(|\beta|^2+|z|^2)^4}. \qquad (3.159)
$$

Moreover, it exist 2 nice polynomials P_1 and P_2 for which

$$
1 - u_\alpha(z) > 0 \qquad (3.160)
$$

and

$$
\det\,(I-\mathbf{D}_\alpha(z)) \geq \frac{\kappa\,(\mathrm{Im}z)^8}{(|\beta|^2+|z|^2)^4} \qquad (3.161)
$$

for each $z \in \mathcal{B}_N$, where $\mathcal{B}_N$ is defined as

$$\mathcal{B}_N = \left\{ z \in \mathbb{C}^+, \ \frac{1}{MN} P_1(|z|) P_2\left(\frac{1}{\text{Im}z}\right) \leq 1 \right\}. \tag{3.162}$$

Finally, for each $z \in \mathcal{B}_N$, it holds that

$$\det\,(I - \mathbf{D}_{t,\alpha}(z)) \geq \frac{\kappa\,(\text{Im}z)^8}{(|\beta|^2 + |z|^2)^4}. \tag{3.163}$$

Proof. To evaluate $\det\,(I - \mathbf{D}(z))$, we use the calculations of the proof of Lemma 3.7. In particular, we have

$$(I - \mathbf{D}(z)) \begin{pmatrix} \text{Im}t(z) \\ \text{Im}zt(z) \end{pmatrix} = \text{Im}z \begin{pmatrix} \frac{1}{M}\text{Tr}RT(z)T^*(z) \\ 0 \end{pmatrix}. \tag{3.164}$$

This implies that

$$1 - u(z) = \frac{\text{Im}z}{\text{Im}t(z)} \cdot \frac{1}{M}\text{Tr}RT(z)T^*(z) + \frac{\text{Im}zt(z)}{\text{Im}t(z)} v(z) \geq \frac{\text{Im}z}{\text{Im}t(z)} \cdot \frac{1}{M}\text{Tr}RT(z)T^*(z).$$

By applying Cramer's rule to (3.164), we obtain that

$$\det\,(I - \mathbf{D}(z)) = \frac{\text{Im}z}{\text{Im}t(z)} \cdot \frac{1}{M}\text{Tr}RT(z)T^*(z)(1 - u(z)) \geq \left(\frac{\text{Im}z}{\text{Im}t(z)} \cdot \frac{1}{M}\text{Tr}RT(z)T^*(z)\right)^2. \tag{3.165}$$

It is clear that $\text{Im}t(z) \leq |t(z)| \leq \frac{1}{M}\text{Tr}R\,(\text{Im}z)^{-1} \leq b\,(\text{Im}z)^{-1}$. Therefore, it holds that $\dfrac{\text{Im}z}{\text{Im}t(z)} \geq \frac{1}{b}\,(\text{Im}t(z))^2$. We now evaluate $\frac{1}{M}\text{Tr}RT(z)T^*(z)$. For this, we remark that

$$\frac{1}{M}\text{Tr}RT(z)T^*(z) = \frac{1}{M}\text{Tr}RT(z)T^*(z)RR^{-1} \geq \frac{1}{b}\frac{1}{M}\text{Tr}(RT(z)T^*(z)R). \tag{3.166}$$

Jensen's inequality implies that $\frac{1}{M}\text{Tr}(RT(z)T^*(z)R) \geq \left|\frac{1}{M}\text{Tr}RT(z)\right|^2 = |t(z)|^2 \geq (\text{Im}\,t(z))^2$. Therefore, the application of Lemma 3.5 to $\beta(z) = t(z)$ implies that

$$\left(\frac{\text{Im}z}{\text{Im}t(z)} \cdot \frac{1}{M}\text{Tr}RT(z)T^*(z)\right)^2 \geq \frac{\kappa\,(\text{Im}z)^8}{(|\beta|^2 + |z|^2)^4}$$

for some nice constants κ and β. (3.159) thus follows from (3.165).

We now establish (3.160) and (3.161), and denote by $\epsilon(z)$ the function $\epsilon(z) = \alpha(z) - s(z)$. Using the equation $s(z) = \frac{1}{M}\text{Tr}RS(z)$, and calculating $\text{Im}\,s(z)$ and $\text{Im}\,zs(z)$, we obtain immediately that

$$(\mathbf{I} - \mathbf{D}_\alpha(z)) \begin{pmatrix} \text{Im}\alpha(z) \\ \text{Im}z\alpha(z) \end{pmatrix} = \text{Im}z \begin{pmatrix} \frac{1}{M}\text{Tr}RS(z)S^*(z) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{Im}\epsilon(z) \\ \text{Im}z\epsilon(z) \end{pmatrix}. \tag{3.167}$$

The first component of (3.167) leads to

$$1 - u_\alpha = \frac{\text{Im}z}{\text{Im}\alpha} \cdot \frac{1}{M}\text{Tr}RSS^* + \frac{\text{Im}\epsilon}{\text{Im}\alpha} + \frac{\text{Im}z\alpha}{\text{Im}\alpha} v_\alpha \geq \frac{\text{Im}z}{\text{Im}\alpha} \cdot \frac{1}{M}\text{Tr}RSS^* + \frac{\text{Im}\epsilon}{\text{Im}\alpha}. \tag{3.168}$$

Using the same arguments as above, we obtain that $\dfrac{1}{M}\text{Tr}RSS^* \geq \frac{1}{b}\,|s(z)|^2 \geq \frac{1}{b}\,(\text{Im}s(z))^2$. As (3.154) holds, we can apply Lemma 3.5 to $\beta(z) = s(z)$ and obtain as above that

$$\frac{\text{Im}z}{\text{Im}s(z)} \cdot \frac{1}{M}\text{Tr}RS(z)S^*(z) \geq \frac{\kappa\,(\text{Im}z)^4}{(|\beta|^2 + |z|^2)^2}$$

for some nice constants β and κ. We remark that $\dfrac{\mathrm{Im}\epsilon}{\mathrm{Im}\alpha} \geq -\frac{|\epsilon|}{\mathrm{Im}\alpha}$. Therefore, by Lemma 3.5 applied to $\beta(z) = \alpha(z)$, it holds that $\dfrac{\mathrm{Im}\epsilon}{\mathrm{Im}\alpha} \geq -\kappa_1|\epsilon|\frac{\beta_1^2+|z|^2}{\mathrm{Im}z}$ for some nice constants κ_1 and β_1. As $|\epsilon(z)| \leq \frac{1}{N^2}Q_1(|z|)Q_2(\frac{1}{\mathrm{Im}z})$ for some nice polynomials Q_1 and Q_2,we obtain that

$$1 - u_\alpha \geq \frac{\mathrm{Im}z}{\mathrm{Im}\alpha} \cdot \frac{1}{M}\mathrm{Tr}RSS^* + \frac{\mathrm{Im}\epsilon}{\mathrm{Im}\alpha} \geq \frac{\mathrm{Im}z}{\mathrm{Im}\alpha} \cdot \frac{1}{M}\mathrm{Tr}RSS^* - \frac{|\epsilon|}{\mathrm{Im}\alpha} \geq \frac{1}{2}\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2} \tag{3.169}$$

if z belongs to the set $\mathcal{B}_{1,N}$ defined by

$$\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2} - \frac{1}{N^2}Q_1(|z|)Q_2(\frac{1}{\mathrm{Im}z})\,\kappa_1\frac{\beta_1^2+|z|^2}{\mathrm{Im}z} \geq \frac{1}{2}\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2}.$$

The set $\mathcal{B}_{1,N}$ is clearly defined in the same way than $\mathcal{B}_N$, but from 2 other nice polynomials $P_{1,1}$ and $P_{2,1}$. Using the Cramer rule, we obtain that det $(\mathbf{I} - \mathbf{D}_\alpha)$ can be written as

$$\det\,(\mathbf{I} - \mathbf{D}_\alpha) = \left(\frac{\mathrm{Im}z}{\mathrm{Im}\alpha} \cdot \frac{1}{M}\mathrm{Tr}RSS^* + \frac{\mathrm{Im}\epsilon}{\mathrm{Im}\alpha}\right)(1-u_\alpha) + \frac{\mathrm{Im}z\epsilon}{\mathrm{Im}\alpha}v_\alpha.$$

Plugging (3.169) in the last equation, we get that the inequality

$$\det\,(\mathbf{I} - \mathbf{D}_\alpha) \geq \left(\frac{1}{2}\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2}\right)^2 - \frac{|z|\,|\epsilon|}{\mathrm{Im}\alpha}v_\alpha$$

holds for each $z \in \mathcal{B}_{1,N}$. As $v_\alpha = \mathcal{O}_z(1)$, we obtain that

$$\left(\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2}\right)^2 - \frac{|z|\,|\epsilon|}{\mathrm{Im}\alpha}v_\alpha \geq \left(\frac{1}{4}\frac{\kappa\ (\mathrm{Im}z)^4}{(|\beta|^2+|z|^2)^2}\right)^2$$

for each $z \in \mathcal{B}_{2,N}$, where $\mathcal{B}_{2,N}$ is defined as $\mathcal{B}_N$ from 2 nice polynomials $P_{1,2}$ and $P_{2,2}$. We put $P_1(|z|) = P_{1,1}(|z|)+P_{1,2}(|z|)$ and $P_2(1/\mathrm{Im}z) = P_{2,1}(1/\mathrm{Im}z)+P_{2,2}(1/\mathrm{Im}z)$, and consider the set $\mathcal{B}_N$ defined by (3.162). It is clear that $\mathcal{B}_N \subset \mathcal{B}_{1,N} \cap \mathcal{B}_{2,N}$, and that (3.160) and (3.161) hold if $z \in \mathcal{B}_N$.

It remains to establish (3.163). For this, we remark that the inequalities

$$\begin{aligned}
|\det\,(\mathbf{I} - \mathbf{D}_{t,\alpha}(z))| &\geq |1-u_{t,\alpha}(z)|^2 - |z|^2|v_{t,\alpha}(z)|^2 \geq (1-|u_{t,\alpha}(z)|)^2 \\
&- |z|v_\alpha(z)\cdot|z|v(z) \geq (1-\sqrt{u(z)u_\alpha(z)})^2 - |z|v_\alpha(z)\cdot|z|v(z) \geq (1-u(z))(1-u_\alpha(z)) \\
&- |z|v_\alpha(z)\cdot|z|v(z) \geq \sqrt{((1-u(z))^2-|z|^2v(z))((1-u_\alpha(z))^2-|z|^2v_\alpha(z))} \\
&= \sqrt{\det\,(\mathbf{I}-\mathbf{D}(z))\det\,(\mathbf{I}-\mathbf{D}_\alpha(z))}
\end{aligned}$$

hold for each $z \in \mathcal{B}_N$. Therefore, (3.163) follows from (3.159) and (3.161). This completes the proof of Lemma 3.15. ■

Solving (3.158), we obtain immediately that it exists 2 nice polynomials Q_1 and Q_2 such that,

$$|s_N(z) - t_N(z)| \leq \frac{1}{MN}Q_1(|z|)Q_2\left(\frac{1}{\mathrm{Im}z}\right)$$

holds for each $z \in \mathcal{B}_N$. If $z \in \mathcal{B}_N^c$, we use the argument in [17]. More precisely, if $z \in \mathcal{B}_N^c$, the inequality $1 < \frac{1}{MN}P_1(|z|)P_2(1/\mathrm{Im}z)$ holds. As $|s_N(z) - t_N(z)| \leq 2\,\frac{1}{M}\mathrm{Tr}R_N\,\frac{1}{\mathrm{Im}z}$ on $\mathbb{C}^+$, we deduce that

$$|s_N(z) - t_N(z)| \leq 2b\frac{1}{MN}P_1(|z|)\frac{P_2(1/\mathrm{Im}z)}{\mathrm{Im}z}$$

for each $z \in \mathcal{B}_N^c$. This, in turn, leads to the conclusion that $s_N(z) - t_N(z) = \mathcal{O}_z(\frac{1}{N^2})$ for each $z \in \mathbb{C}^+$. This establishes (3.155) and $\frac{1}{M}\mathrm{Tr}(T_N(z) - S_N(z)) = \mathcal{O}_z(\frac{1}{N^2})$ as expected. This completes the proof of Proposition 3.11. ■

We now follow [8] and [17] and use the following Lemma.

Lemma 3.16. *Let ϕ be a compactly supported real valued smooth function defined on $\mathbb{R}^+$, i. e. $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^+, \mathbb{R}^+)$. Then,*

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} - \int_{\mathcal{S}_N} \phi(\lambda) d\mu_N(\lambda) = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Proof. Due to Proposition 2.1 we can write

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} = \frac{1}{\pi}\lim_{y\downarrow 0}\mathrm{Im}\left\{\int_{\mathbb{R}^+}\phi(x)\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}Q(x+iy)\right\}dx\right\}$$

as well as

$$\int_{\mathcal{S}_N}\phi(\lambda)d\mu_N(\lambda) = \frac{1}{\pi}\lim_{y\downarrow 0}\mathrm{Im}\left\{\int_{\mathbb{R}^+}\phi(x)\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}T(x+iy)\right\}dx\right\}$$

Using Proposition 3.11, we obtain

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} - \int_{\mathcal{S}_N}\phi(\lambda)d\mu_N(\lambda)$$
$$= \frac{1}{N^2}\frac{1}{\pi}\lim_{y\downarrow 0}\mathrm{Im}\left\{\int_{\mathbb{R}^+}\phi(x) r_N(x+iy)dx\right\}. \quad (3.170)$$

Since the function $r_N(z) = \mathcal{O}_z(1)$, we can use the result which was proved in [7, Section 3.3] and obtain

$$\limsup_{y\downarrow 0}\left|\int_{\mathbb{R}^+}\phi(x) r_N(x+iy)dx\right| \le \kappa$$

for some nice constant κ. This and (3.170) complete the proof. ■

In order to establish Theorem 3.4, we introduce a function $\phi \in \mathcal{C}_c^\infty$ such that $0 \le \phi(\lambda) \le 1$ and

$$\phi(\lambda) = \begin{cases} 1, \text{ for } \lambda \in [\kappa_1, \kappa_2], \\ 0, \text{ for } \lambda \in \mathbb{R} - (\kappa_1 - \epsilon, \kappa_2 + \epsilon). \end{cases}$$

Since for N large enough $(\kappa_1 - \epsilon, \kappa_2 + \epsilon) \cap \mathcal{S}_N = \varnothing$ then $\int_{\mathcal{S}_N}\phi(\lambda)d\mu_N(\lambda) = 0$ and according to Lemma 3.16

$$\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Now we show that

$$\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} = \mathcal{O}\left(\frac{1}{N^4}\right).$$

For this we use again the Poincare-Nash inequality

$$\mathbf{Var}\{\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\} \le \sum \mathbb{E}\Big\{\left(\frac{\partial \mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)}{\partial \overline{W}_{i_1,j_1}^{m_1}}\right)^* \mathbb{E}\{W_{i_1,j_1}^{m_1}\overline{W}_{i_2,j_2}^{m_2}\}$$
$$\times \frac{\partial \mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)}{\partial \overline{W}_{i_2,j_2}^{m_2}}\Big\} + \sum \mathbb{E}\left\{\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial W_{i_1,j_1}^{m_1}}\mathbb{E}\{W_{i_1,j_1}^{m_1}\overline{W}_{i_2,j_2}^{m_2}\}\left(\frac{\partial \mathrm{Tr}\phi(WW^*)}{\partial W_{i_2,j_2}^{m_2}}\right)^*\right\}.$$

We only evaluate the first term of the r.h.s. of the inequality, denoted by ψ, because the second is similar. For this we write first

$$\frac{\partial \mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)}{\partial \overline{W}_{i_1,j_1}^{m_1}} = \mathrm{Tr}\left(\phi'(W_f W_p^* W_p W_f^*)\frac{\partial W_f W_p^* W_p W_f^*}{\partial \overline{W}_{i_1,j_1}^{m_1}}\right)$$
$$= \begin{cases} 1 \le i_1 \le L, \ (W_p W_f^* \phi'(W_f W_p^* W_p W_f^*) W_f)_{i_1 j_1}^{m_1}, \\ L+1 \le i_1 \le 2L, \ (\phi'(W_f W_p^* W_p W_f^*) W_f^* W_f W_p)_{i_1 - L, j_1}^{m_1}. \end{cases}$$

Plugging this into (3.6) we obtain

$$\psi = \sum_{i_1,i_2=1}^{L} \sum_{j_1,j_2,m_1,m_2} \Big(\frac{1}{N}\mathbb{E}\Big\{ \big(W_pW_f^*\phi'(W_fW_p^*W_pW_f^*)W_f\big)^{*m_1}_{i_1j_1} R_{m_1m_2}\delta_{i_1+j_1,i_2+j_2}$$
$$\times \big(W_pW_f^*\phi'(W_fW_p^*W_pW_f^*)W_f\big)^{m_2}_{i_2,j_2} \Big\} + \frac{1}{N}\mathbb{E}\Big\{ \big(\phi'(W_fW_p^*W_pW_f^*)W_fW_p^*W_p\big)^{*m_1}_{i_1j_1}$$
$$\times R_{m_1m_2}\delta_{i_1+j_1,i_2+j_2} \big(\phi'(W_fW_p^*W_pW_f^*)W_fW_p^*W_p\big)^{m_2}_{i_2,j_2} \Big\}\Big).$$

Following the proof of Lemma 3.1, we obtain

$$\mathbf{Var}\{\mathrm{Tr}\phi(W_fW_p^*W_pW_f^*)\} \le \frac{C}{N}\mathbb{E}\{\mathrm{Tr}W_f^*\phi'(W_fW_p^*W_pW_f^*)W_fW_p^*W_pW_f^*$$
$$\times \phi'(W_fW_p^*W_pW_f^*)W_f\} + \frac{C}{N}\mathbb{E}\{\mathrm{Tr}W_fW_p^*W_pW_p^*W_pW_f^* \left(\phi'(W_fW_p^*W_pW_f^*)\right)^2\}. \quad (3.171)$$

To evaluate the first term ψ_1 of the r.h.s of (3.171) we denote $\eta(\lambda) = (\phi'(\lambda))^2\lambda$ and write

$$\frac{1}{N}\mathbb{E}\left\{\mathrm{Tr}W_f^*\phi'(W_fW_p^*W_pW_f^*)W_fW_p^*W_pW_f^*\phi'(W_fW_p^*W_pW_f^*)W_f\right\}$$
$$\le \frac{1}{N}\mathbb{E}\left\{\|W_f\|^2\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\right\}.$$

We recall that (3.5) implies that $\|W_f\|^2 \le b\|W_{iid}\|^2$. Therefore, it holds that

$$\psi_1 \le \frac{\kappa}{N}\mathbb{E}\{\|W_{iid}\|^2\mathbf{1}_{\|W_{iid}\|\le(1+\sqrt{c_*})^2+\delta}\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\}$$
$$+ \frac{\kappa}{N}\mathbb{E}\{\|W_{iid}\|^2\mathbf{1}_{\|W_{iid}\|>(1+\sqrt{c_*})^2+\delta}\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\}$$
$$\le \frac{\kappa}{N}\mathbb{E}\{\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\} + \kappa\mathbb{E}^{1/2}\{\|W_{iid}\|^4\mathbf{1}_{\|W_{iid}\|>(1+\sqrt{c_*})^2+\delta}\}$$
$$\times \mathbb{E}^{1/2}\left\{\left(\frac{1}{N}\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\right)^2\right\}.$$

Lemma 3.16 implies that $\frac{1}{N}\mathbb{E}\{\mathrm{Tr}(\eta(W_fW_p^*W_pW_f^*))\} = \mathcal{O}(N^{-2})$. Throughout the proof of Lemma 3.1, we get that $\mathbb{E}\|W_{iid}\|^4\mathbf{1}_{\|W_{iid}\|>(1+\sqrt{c_*})^2+\delta} = \mathcal{O}(N^{-k})$ for all k. Since function $\phi' \in \mathcal{C}_c^\infty$, there exists a nice constant κ such that $|\phi'(\lambda)| < \kappa$ for all λ and $\phi'(\lambda) = 0$ for all $\lambda > b + 2\epsilon$. We deduce from this that it exists a nice constant κ such that $\|\eta(W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*)\| < \kappa$ for each N. From what about we conclude that $\psi_1 = \mathcal{O}(N^{-2})$.
As for the second term (ψ_2) of the r.h.s of (3.171), we write

$$\psi_2 = \frac{\kappa}{N}\mathbb{E}\left\{\mathrm{Tr}W_p^*W_pW_p^*W_pW_f^*\left(\phi'(W_fW_p^*W_pW_f^*)\right)^2W_f\right\}$$
$$\le \kappa\mathbb{E}\left\{\|W_p\|^2\frac{1}{N}\mathrm{Tr}\left(\phi'(W_fW_p^*W_pW_f^*)\right)^2W_fW_p^*W_pW_f^*\right\}.$$

It is easy to see that ψ_2 can be evaluated as ψ_1, leading to the conclusion that $\psi_2 = \mathcal{O}(N^{-2})$. Therefore, we have checked that

$$\mathbf{Var}\{\mathrm{Tr}\phi(W_fW_p^*W_pW_f^*)\} = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Now we can complete the proof of Theorem 3.4 as in [8]. For this we apply the classical Markov inequality and combine what above

$$\mathbf{P}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*) > \frac{1}{N^{4/3}}\right\} \leq N^{8/3}\mathbb{E}\left\{\left(\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right)^2\right\}$$

$$= N^{8/3}\left(\mathbf{Var}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\} + \left(\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*)\right\}\right)^2\right)$$

$$= \mathcal{O}\left(\frac{1}{N^{4/3}}\right).$$

Applying Borel-Cantelli lemma, we obtain that almost surely, the inequality

$$\frac{1}{ML}\mathrm{Tr}\phi(W_f W_p^* W_p W_f^* *) \leq \frac{1}{N^{4/3}}$$

holds for each N large enough. By the very definition of function ϕ, the number of eigenvalues of matrix $W_f W_p^* W_p W_f^*$ lying in the interval $[\kappa_1, \kappa_2]$ is upper bounded by $\mathrm{Tr}\phi(W_f W_p^* W_p W_f^*) \leq \frac{1}{N^{1/3}}$. Since this number of eigenvalues is an integer, we conclude that with probability one there is no eigenvalues in the interval $[\kappa_1, \kappa_2]$ for each N large enough. ■

We finally illustrate the above results by the following numerical experiment. M, N, L are given by $M = 500$, $N = 1500$ and $L = 2$ so that $c_N = 2/3$. The eigenvalues of matrix R_N are defined by $\lambda_{k,N} = 1/2 + \frac{\pi}{4}\cos\left(\frac{\pi(k-1)}{2M}\right)$ for $k = 1, \ldots, M$. Matrix R_N verifies $\frac{1}{M}\mathrm{Tr}(R_N) \simeq 1$. Fig. 3.3 represents the histogram of the eigenvalues of a realization of $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ as well as the graph of the density $g_N(x)$. We notice that the histogram and the graph of g_N are in accordance, and that, as expected, no eigenvalue of $W_{f,N} W_{p,N}^* W_{p,N} W_{f,N}^*$ lies outside the support of g_N.

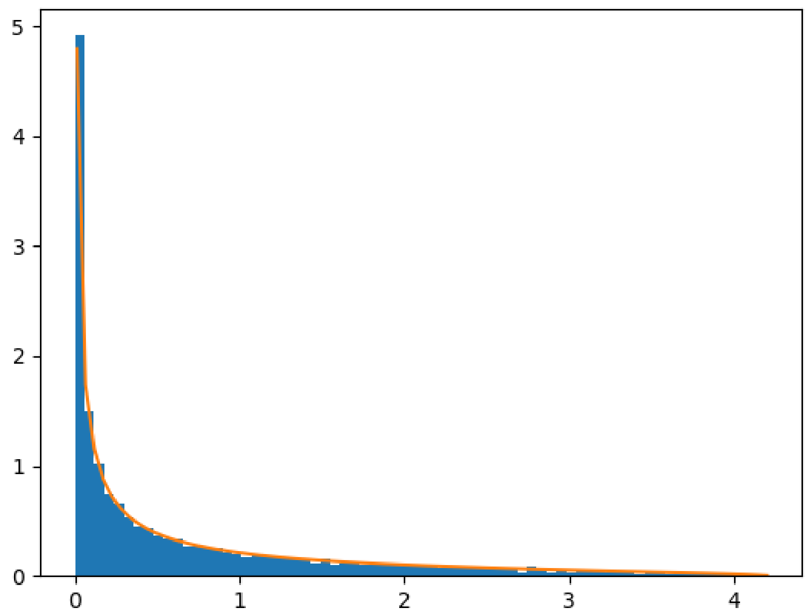

FIGURE 3.3 – Histogram of the eigenvalues and graph of $g_N(x)$ for $M = 500, N = 1500, L = 2$.

3.8 Recovering the behaviour of the empirical eigenvalue distribution $\hat{\nu}_N$ using free probability tools

The purpose of this paragraph is to show that it is possible to use free probability tools in order to characterize the limiting behaviour of the empirical eigenvalue distribution $\hat{\nu}_N$ of matrix $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$. As this thesis is not focused on these kind of approach, we present briefly the following results and leave the details to the reader.

The free probability approach is based on the following observations :

- Up to the zero eigenvalue, the eigenvalues of $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ coincide with the eigenvalues of $W_{f,N}^*W_{f,N}W_{p,N}^*W_{p,N}$
- The matrices $W_{f,N}^*W_{f,N}$ and $W_{p,N}^*W_{p,N}$ are almost surely asymptotically free. Therefore, the eigenvalue distribution of $W_{f,N}^*W_{f,N}W_{p,N}^*W_{p,N}$ converges towards the free multiplicative convolution product of the limit distributions of $W_{f,N}^*W_{f,N}$ and $W_{p,N}^*W_{p,N}$. These two distributions appear to coincide both with the limit distribution of the well known random matrix model $\frac{1}{N}X_N^*(I_L \times R_N)X_N$ where X_N is a $ML \times N$ complex Gaussian random matrix with unit variance i.i.d. entries.

In the following, we follow the definitions of asymptotic freeness provided in [22] (see in particular section 4.3) which need the existence of certain limit distributions. This is in contrast with the approach developed in the previous sections more focused on the behaviour of deterministic equivalents. We however mention that more recent free probability works (see e.g. [36] and the references therein, [6]) allow to avoid the introduction of limit distributions, and would allow to recover the previous results on the deterministic equivalent ν_N of $\hat{\nu}_N$.

In order to be in accordance with [22], we thus formulate in this section the following assumption :

Assumption A-1: *The empirical eigenvalue distribution $\omega_N = \frac{1}{M}\sum_{k=1}^{M}\delta_{\lambda_{k,N}}$ of matrix R_N converges towards a limit distribution ω.*

We remark that hypothesis 2.2 implies that ω is compactly supported. Moreover, it can be shown that measures $(\mu_N)_{N\geq 1}$ and $(\nu_N)_{N\geq 1}$ both converge weakly towards limits denoted μ and ν in this section. We also notice that Lemma 3.14 implies that μ and ν are compactly supported. It is also easily checked that the Stieltjes transform $t(z)$ of μ verifies the equation

$$t(z) = -\frac{1}{z}\int_{\mathbb{R}^+} \frac{\tau\, d\omega(\tau)}{1 + \dfrac{c_*\tau t(z)}{1 - zc_*^2 t^2(z)}}, \tag{3.172}$$

while the Stieltjes transform t_ν of ν is given by

$$t_\nu(z) = -\frac{1}{z} - \frac{c_* t(z)^2}{1 - z(c_* t(z))^2}. \tag{3.173}$$

We recall that c_* represents the limit of $c_N = \frac{ML}{N}$. In the following, we establish that (3.172) and (3.173) can be obtained using free probability technics.

Before going further, we first recall the main useful definitions introduced in [22].

Definition 1. *Consider a finite family of sequences of $N \times N$ possibly random matrices $((X_{i,N})_{N\geq 1})_{i=1,\dots,r}$. Then $(X_{i,N})_{i=1,\dots,r}$ is said to have an almost sure joint limit if for each non commutative polynomial $P(x_1,\dots,x_r)$ in r indeterminates, then $\frac{1}{N}\mathrm{Tr}P(X_{1,N},\dots,X_{r,N})$ converges almost surely towards $\mu(P)$ where μ is a deterministic distribution defined on the set of all non commutative polynomials in r indeterminates (i.e. μ is a linear form such that $\mu(1) = 1$).*

We remark that if $r = 1$ and $(X_{1,N})_{N\geq 1}$ are Hermitian matrices, the above condition is equivalent to the existence of a limit empirical eigenvalue distribution.

Definition 2. *Consider p families $(X_{i,N}^{(1)})_{i=1,\dots,r_1},\dots,(X_{i,N}^{(p)})_{i=1,\dots,r_p}$ of $N \times N$ possibly random matrices. Then, $X^{(1)},\dots,X^{(p)}$ are said to be almost surely asymptotically free if the 2 following conditions hold :*

— *For each* $q = 1, \ldots, p$, $(X^{(q)}_{i,N})_{i=1,\ldots,r_q}$ *has an almost sure joint limit*
— $\forall m$, $i_1, \cdots, i_m \in \{1, 2, \ldots, p\}$ *with* $i_1 \neq i_2 \neq \cdots \neq i_m$, *and for each non commutative polynomials* $(P_j)_{j=1,\ldots,m}$ *in* $(r_{i_j})_{j=1,\ldots,m}$ *indeterminates such that* $\frac{1}{N}\mathrm{Tr}(P_j(X^{i_j}_{1,N}, \ldots, X^{i_j}_{r_{i_j},N})) \to 0$ *a.s. it holds that*

$$\frac{1}{N}\mathrm{Tr}(P_1(X^{i_1}_{1,N}, \ldots, X^{i_1}_{r_{i_1},N}) \cdots P_m(X^{i_m}_{1,N}, \ldots, X^{i_m}_{r_{i_m},N})) \to 0 \quad a.s.$$

We remark that when each family $X^{(q)}$ is reduced to a single sequence $(X^{(q)}_N)_{N\geq 1}$ of $N \times N$ Hermitian, or similar to hermitian matrices [1], the almost sure freeness of $X^{(1)}, \ldots, X^{(p)}$ holds if

Definition 3. — *For each* $q = 1, \ldots, p$, $(X^{(q)}_N)_{N\geq 1}$ *has a limit eigenvalue distribution*
— $\forall m$, $i_1, \cdots, i_m \in \{1, 2, \ldots, p\}$ *with* $i_1 \neq i_2 \neq \cdots \neq i_m$, *and for each polynomials* $(P_j)_{j=1\cdots m}$ *in one indeterminate such that* $\frac{1}{N}\mathrm{Tr}(P_j(X^{i_j}_N)) \to 0$ *a.s. it holds that*

$$\frac{1}{N}\mathrm{Tr}(P_1(X^{(i_1)}_N)P_2(X^{(i_2)}_N) \cdots P_m(X^{(i_m)}_N)) \to 0 \quad a.s. \tag{3.174}$$

We also recall the definition of the S transform of a probability measure, and recall that the S transform of the free multiplicative convolution product of two probability measures is the product of their S transforms.

Definition 4. *Given a compactly supported probability measure* μ *carried by* $\mathbb{R}^+$, *we define* $\psi_\mu(z)$ *as the formal power series defined by*

$$\psi_\mu(z) = \sum_{k\geq 1} z^k \int t^k d\mu(t) = \int \frac{zt}{1-zt} d\mu(t) \tag{3.175}$$

Let χ_μ *be the unique function analytic in a neighbourhood of zero, satisfying*

$$\chi_\mu(\psi_\mu(z)) = z \tag{3.176}$$

for $|z|$ *small enough. Then, we define the* S *transform of* μ *as the function* $S_\mu(z)$ *defined in a neighbourhood of zero by*

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z}. \tag{3.177}$$

Moreover, if μ_1 *and* μ_2 *are two compactly supported probability measures carried by* $\mathbb{R}^+$, *the S-transform* $S_{\mu_1 \boxtimes \mu_2}$ *of* $\mu_1 \boxtimes \mu_2$ *satisfies*

$$S_{\mu_1 \boxtimes \mu_2} = S_{\mu_1} S_{\mu_2}. \tag{3.178}$$

We are now in position to state the main result of this section.

Proposition 3.12. *Matrices* $W^*_{f,N} W_{f,N}$ *and* $W^*_{p,N} W_{p,N}$ *are almost surely asymptotically free.*

Proof. We first notice that it possible to replace matrices W_f and W_p by finite rank perturbations because the very definition of almost sure asymptotic freeness is not affected by finite rank perturbations. We thus exchange W_p and W_f by $\tilde{W}_p = \frac{1}{\sqrt{N}}\tilde{Y}_p$ and $\tilde{W}_f = \frac{1}{\sqrt{N}}\tilde{Y}_f$ where $\tilde{Y}_p$ and $\tilde{Y}_f$ are defined by

$$\tilde{Y}_p = \begin{pmatrix} y_1 & \cdots & \cdots & \cdots & \cdots & \cdots & y_N \\ y_2 & \cdots & \cdots & \cdots & \cdots & y_N & y_1 \\ y_3 & \cdots & \cdots & \cdots & y_N & y_1 & y_2 \\ \vdots & \cdots & \cdots & \iddots & \iddots & \iddots & \vdots \\ y_L & \cdots & y_N & y_1 & y_2 & \cdots & y_{L-1} \end{pmatrix},$$

1. in the sense that $X^{(q)}_N = U^{(q)}_N H^{(q)}_N (U^{(q)}_N)^{-1}$ for some $N \times N$ Hermitian matrix $H^{(q)}_N$

$$\tilde{Y}_f = \begin{pmatrix} y_{L+1} & \cdots & \cdots & \cdots & \cdots & \cdots & y_N & y_1 & \cdots & y_L \\ y_{L+2} & \cdots & \cdots & \cdots & \cdots & y_N & y_1 & \cdots & y_L & y_{L+1} \\ y_{L+3} & \cdots & \cdots & \cdots & y_N & y_1 & \cdots & y_L & y_{L+1} & y_{L+2} \\ \vdots & \cdots & \cdots & \iddots & \iddots & \cdots & \iddots & \iddots & \iddots & \vdots \\ y_{2L} & \cdots & y_N & y_1 & \cdots & y_L & y_{L+1} & y_{L+2} & \cdots & y_{2L-1} \end{pmatrix}.$$

In other words, vectors $y_{N+1}, \dots, y_{N+L-1}, \dots, y_{N+2L-1}$ are replaced by vectors $y_1, \dots, y_{L-1}, \dots, y_{2L-1}$. In order to simplify the notations, we still denote the above finite rank modifications by Y_p, Y_f, W_p, W_f. We define the $N \times N$ matrix Π and the $M \times N$ matrix Y by

$$\Pi = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \text{ and } Y = (y_1, y_2, \dots, y_N)$$

and rewrite Y_p (and Y_f respectively) as

$$Y_p = \begin{pmatrix} Y \\ Y\Pi \\ \vdots \\ Y\Pi^{L-1} \end{pmatrix}, \quad Y_f = \begin{pmatrix} Y\Pi^L \\ Y\Pi^{L+1} \\ \vdots \\ Y\Pi^{2L-1} \end{pmatrix}.$$

This allows us to obtain the useful expression for $W_p^* W_p$ and $W_f^* W_f$

$$W_p^* W_p = \sum_{k=0}^{L-1} \Pi^{*k} \left(\frac{Y^* Y}{N} \right) \Pi^k, \quad (3.179)$$

$$W_f^* W_f = \sum_{k=L}^{2L-1} \Pi^{*k} \left(\frac{Y^* Y}{N} \right) \Pi^k. \quad (3.180)$$

Since $N^{-1} Y^* Y$ can be written as $N^{-1} Y_{iid}^* R_N Y_{iid}$, where Y_{iid} has i.i.d. Gaussian entries, the hermitian matrix $N^{-1} Y^* Y$ is unitarily invariant. Moreover, Assumption 1 implies that $N^{-1} Y^* Y$ has a limit distribution while it is easily checked that the family $\{I, \Pi^*, \Pi, \dots, \Pi^{*2L-1}, \Pi^{2L-1}\}$ has the same property. This and Theorem 4.3.5 in [22] leads to the conclusion that $Y^* Y / N$ and $\{I, \Pi^*, \Pi, \dots, \Pi^{*2L-1}, \Pi^{2L-1}\}$ are almost surely asymptotically free. Proposition 3.12 thus appears to be an immediate consequence of the following Lemma adapted from Lemma 6 in [15]. In order to make the connections between Lemma 3.17 and Lemma 6 in [15], we use nearly the same notations than in [15] in the following statement.

Lemma 3.17. *We consider a sequence of $N \times N$ Hermitian random matrices $(X^N)_{N \geq 1}$ and $N \times N$ deterministic matrices $U_1^N, W_1^N, \dots, U_m^N, W_m^N$ such that X_N and $\{U_1^N, W_1^N, \dots, U_m^N, W_m^N\}$ are almost surely asymptotically free. Then, if $U_1^N, W_1^N, \dots, U_m^N, W_m^N$ satisfy*

$$U_i^N W_i^N = W_i^N U_i^N = I_N \quad (3.181)$$

for each $i = 1, \dots, m$ as well as $\frac{1}{N} \mathrm{Tr}(U_i^N W_j^N) = \delta_{i-j}$ for all $i, j = 1 \dots m$, then the random matrices $U_1^N X^N W_1^N, \dots, U_m^N X^N W_m^N$ are almost surely asymptotically free.

Proof. We prove Lemma 3.17 by following step by step the proof from [15]. For simplicity we omit index N below. Due to (3.181) we have $W_i = U_i^{-1}$ so that matrices $(U_i X W_i)_{i=1,\dots,m}$ are similar to the Hermitian matrix X. We have thus to verify the 2 items of Definition 3. The first item is obvious. To check condition (3.174), we consider any k, indexes $i_1, \cdots, i_k$ with $i_1 \neq \cdots \neq i_k$ and polynomials P_j such that $\frac{1}{n} \mathrm{Tr}(P_j(U_{i_j} X W_{i_j})) \to 0$ a.s. Using again (3.181) it is clear that $P_j(U_{i_j} X W_{i_j}) = U_{i_j} P_j(X) W_{i_j}$ and, as a consequence, $\frac{1}{n} \mathrm{Tr}(P_j(X)) \to 0$ a.s. We define η_N as

$$\eta_N = \frac{1}{N} \mathrm{Tr}(P_1(U_{i_1} X W_{i_1}) P_2(U_{i_2} X W_{i_2}) \cdots (U_{i_k} X W_{i_k})) =$$

$$\frac{1}{N} \mathrm{Tr}(U_{i_1} P_1(X) W_{i_1} U_{i_2} P_2(X) W_{i_2} \cdots U_{i_k} P_k(X) W_{i_k}) = \frac{1}{N} \mathrm{Tr} \left(\prod_{j=1}^{k} W_{i_{j-1}} U_{i_j} P_j(X) \right),$$

where $i_0 = i_k$. If $i_1 \neq i_k$ then by assumption $\frac{1}{n}\mathrm{Tr}(W_{i_{j-1}}U_{i_j}) = 0$ for $j = 1, \ldots, m$. As we also have $\frac{1}{n}\mathrm{Tr}(P_j(X)) \to 0$ a.s, the almost sure asymptotic freeness of X and $\{U_1, W_1, \cdots, U_m, W_m\}$ leads to the conclusion that $\eta_N \to 0$ a.s. In the case when $i_1 = i_k$ we have $W_{i_k}U_{i_1} = I_N$ and the same conclusion holds. □

By taking $X = \frac{YY^*}{N}$, $U_i = \Pi^{*i-1}$ and $W_i = \Pi^{i-1}$, Lemma 3.17 gives us immediately that $\frac{Y^*Y}{N}$, $\Pi^*(\frac{Y^*Y}{N})\Pi$, $\ldots, \Pi^{*2L-1}(\frac{Y^*Y}{N})\Pi^{2L-1}$ are almost surely asymptotically free. Using the expression (3.179, 3.180) of $W_p^*W_p$ and $W_f^*W_f$, we obtain that $W_p^*W_p$ and $W_f^*W_f$ are almost surely asymptotically free. ■

We also deduce that the limit distributions of $W_p^*W_p$ and $W_f^*W_f$ both coincide with the additive free convolution product of L copies of the well known limit distribution of $\frac{Y^*Y}{N}$. It is easily seen that the Stieljes transform, denoted $t_{MP}(z)$ in the following, of this free addditive convolution product is solution of the familiar equation

$$t_{MP}(z) = -\frac{1}{z - c_* \int \frac{\tau\omega(d\tau)}{1 + \tau t_{MP}(z)}}. \tag{3.182}$$

In the following, we denote by μ_{MP} the corresponding probability measure. It is clear that (3.182) coincides with the equation verified by the Stieltjes transform of the limit eigenvalue distribution of the random matrix $\frac{1}{N}X_N^*(I_L \times R_N)X_N$ where X_N is a $ML \times N$ complex Gaussian random matrix with unit variance i.i.d. entries. We note that this result could also be easily obtained using the Gaussian technics developed in [32] in the case where R_N is reduced to a multiple of I_M.

According to Proposition 3.12, the limit eigenvalue distribution of $W_{f,N}^*W_{f,N}W_{p,N}^*W_{p,N}$ is $\mu_{MP} \boxtimes \mu_{MP}$. In the following, we denote by $\tilde{\nu}$ this measure and by $\tilde{f}(z)$ its Stieltjes transform. To find an equation satisfied by $\tilde{f}(z)$, we use (3.178). (3.177) and (3.178) give us immediately

$$\chi_{\tilde{\nu}}(z) = \frac{1+z}{z}\chi_{MP}^2(z).$$

By replacing here z with $\psi_{\tilde{\nu}}(z)$ and taking into account (3.176) we obtain

$$z = \frac{1 + \psi_{\tilde{\nu}}(z)}{\psi_{\tilde{\nu}}(z)}\chi_{MP}^2(\psi_{\tilde{\nu}}(z)). \tag{3.183}$$

We notice that by definition (3.175), we have

$$\psi_{\tilde{\nu}}(z) = \int \frac{zt}{1-zt}d\tilde{\nu}(t) = \int \frac{d\tilde{\nu}(t)}{1-zt} - 1 = -\frac{1}{z}\tilde{f}\left(\frac{1}{z}\right) - 1. \tag{3.184}$$

Putting this into (3.183) and replacing z with $\frac{1}{z}$ give us

$$\frac{z^2\tilde{f}(z)}{1 + z\tilde{f}(z)}\chi_{MP}^2\left(\psi_{\tilde{\nu}}\left(\frac{1}{z}\right)\right) = 1.$$

From this, it is straightforward to obtain the expression of $\tilde{f}(z)$. For more convenience, we introduce the function $g(z) = \chi_{MP}(\psi_{\tilde{\nu}}(z^{-1}))$ which is analytic in the neighbourhood of infinity. It holds that

$$\tilde{f}(z) = \left(z^2g^2(z) - z\right)^{-1}. \tag{3.185}$$

It remains to determine $g(z)$. For this we use (3.184) for ψ_{MP}, t_{MP} and replace z with $\chi_{MP}(z)$. Then (3.176) gives

$$z = -1 - \frac{1}{\chi_{MP}(z)}t_{MP}\left(\frac{1}{\chi_{MP}(z)}\right) \Rightarrow t_{MP}(\chi_{MP}^{-1}(z)) = -(1+z)\chi_{MP}(z).$$

To obtain the equation for χ_{MP} it is sufficient to use the above expression of $t_{MP}(\chi_{MP}^{-1}(z))$, and to plug it in (3.182) with $z = \chi_{MP}^{-1}(z)$. Therefore, we obtain that

$$(1+z)\chi_{MP}(z) = \frac{1}{\dfrac{1}{\chi_{MP}(z)} - c_* \int \dfrac{\tau d\omega_(\tau)}{1-\tau(1+z)\chi_{MP}(z)}}.$$

After simple algebra we get that

$$\frac{z}{(1+z)\chi_{MP}(z)} = c_* \int \frac{\tau d\omega_(\tau)}{1-\tau(1+z)\chi_{MP}(z)}.$$

We finally replace z by $\psi_{\tilde{\nu}}(z^{-1})$ in the above equation. Using (3.183), it is easy to see that the l.h.s. is equal to $zg(z)$. To evaluate the r.h.s., we use again (3.183) and obtain that $\psi_{\tilde{\nu}}(z^{-1}) = zg^2(z)(1-zg^2(z))^{-1}$, and that

$$g(z) = \frac{1}{z} \int_{\mathbb{R}^+} \frac{c_* \tau \, d\omega(\tau)}{1 - \dfrac{\tau g(z)}{1-zg^2(z)}}. \tag{3.186}$$

We recall that $t(z)$ is solution of the equation

$$t(z) = -\frac{1}{z} \int \frac{\tau \omega(d\tau)}{1 + \dfrac{c_* \tau t(z)}{1 - zc_*^2 t^2(z)}}. \tag{3.187}$$

The equations (3.186) and (3.187) are identical up to factor $-c_*$. Since it can be shown that Eq. (3.187) has a unique solution on the set of Stieltjes transforms, we obtain that $g(z) = -c_* t(z)$. Therefore, (3.185) leads to the equation

$$\tilde{f}(z) = -\frac{1}{z\left[1 - z(c_* t(z))^2\right]}.$$

The Stieltjes transform of the limit eigenvalue distribution of $W_f W_p^* W_p W_f^*$ is clearly equal to $\frac{1}{c_*}\left(\tilde{f}(z) + \frac{1-c_*}{z}\right)$. Using the expression (3.173) of $t_\nu(z)$, we obtain immediately that

$$\frac{1}{c_*}\left(\tilde{f}(z) + \frac{1-c_*}{z}\right) = t_\nu(z).$$

We have thus proved that the limit eigenvalue distribution of $W_f W_p^* W_p W_f^*$ can be evaluated using free probability technics.

Chapitre 4

In the presence of signal.

In this chapter, we assume that signal $(u_n)_{n\in\mathbb{Z}}$ is present, and evaluate its influence on the eigenvalues and eigenvectors of matrix $\frac{Y_f Y_p^*}{N}\left(\frac{Y_f Y_p^*}{N}\right)^*$. For this, we use a classical approach based on the observation that matrix $\frac{Y_f Y_p^*}{N}$ is a finite rank perturbation of matrix $\frac{V_f V_p^*}{N}$ due to the noise $(v_n)_{n\in\mathbb{Z}}$. It will be assumed that for each N large enough, the support $\mathcal{S}_N$ of the support of measure μ_N associated to $t_N(z)$ is reduced to the single interval $\mathcal{S}_N = [0, x_{N,+}]$, see paragraph 4.2 for more details on the assumptions that are needed to establish solid mathematical results.

4.1 Signal model and first assumptions

We recall that the useful signal $(u_n)_{n\in\mathbb{Z}}$ is generated by the minimal state-space representation (1.2). As M is supposed to increase towards $+\infty$, it is first necessary to precise how the parameters of (1.2) depend on M. We formulate the following assumptions :

Assumption A-2:
- *$(\omega_n)_{n\in\mathbb{Z}}$ is a K–dimensional white noise sequence such that $\mathbb{E}(\omega_n\omega_n^*) = I_K$, and which is independent of M and N*
- *The dimension P of the state-space does not scale with M and N and matrices A and B are independent of M and N.*
- *Matrices $C = C_N$ and $D = D_N$ depend of M and thus on N, and are supposed to verify*

$$\sup_N \|C_N\| < +\infty, \ \sup_N \|D_N\| < +\infty \tag{4.1}$$

We assume moreover from now on that $L \geq P$. As a consequence of Assumptions 2, the P–dimensional Markovian signal $(x_n)_{n\in\mathbb{Z}}$ is independent of M and N. We define matrix $\mathcal{H}_N$ as the $ML \times KL$ block-Toeplitz matrix defined by

$$\mathcal{H}_N = \begin{pmatrix} D_N & 0 & \dots & \dots & 0 \\ C_N B & D_N & 0 & \ddots & 0 \\ \vdots & C_N B & \ddots & \ddots & \vdots \\ C_N A^{L-3} B & \ddots & \ddots & \ddots & \vdots \\ C_N A^{L-2} B & C_N A^{L-3} B & \ddots & C_N B & D_N \end{pmatrix} \tag{4.2}$$

Then, it is easy to check that the ML–dimensional vector $u_n^L = (u_n^T, \dots, u_{n+L-1}^T)^T$ can be written as

$$u_n^L = (\mathcal{O}_N, \mathcal{H}_N) \begin{pmatrix} x_n \\ \omega_n^L \end{pmatrix} \tag{4.3}$$

where ω_n^L is defined as u_n^L and where we recall that the observability matrix $\mathcal{O}_N$ is defined by (1.7). We formulate the following assumption :

Assumption A-3: *The rank $r \leq P + KL$ of matrix $(\mathcal{O}_N, \mathcal{H}_N)$ remains constant for N large enough.*

As $L \geq P$, the rank of matrix $\mathcal{O}_N$ is equal to P so that $r \geq P$. The covariance matrix $R^L_{u,N} = \mathbb{E}(u^L_n u^{*L}_n)$ is given by

$$R^L_{u,N} = (\mathcal{O}_N, \mathcal{H}_N) \begin{pmatrix} R_x & 0 \\ 0 & I_{KL} \end{pmatrix} (\mathcal{O}_N, \mathcal{H}_N)^*$$

where $R_x = \mathbb{E}(x_n x_n^*)$ coincides with

$$R_x = \sum_{k=0}^{\infty} A^k B B^* A^{*k}$$

R_x is positive definite because the minimality of the state-space representation (1.2) of u implies that the pair (A, B) is commandable. We deduce from this and from Assumption 3 that $\text{Rank}(R^L_{u,n}) = r$ for each N large enough. In the following, we denote by

$$R^L_{u,N} = \Theta_N \Delta^2_N \Theta^*_N \tag{4.4}$$

the eigenvalue / eigenvector decomposition of $R^L_{u,N}$ where $\Delta^2_N = \text{Diag}(\delta^2_{1,N}, \ldots, \delta^2_{r,N})$ and where Θ_N is the $ML \times r$ orthogonal matrix corresponding to the eigenvectors of $R^L_{u,N}$.

In the following, we denote by $X_{1,N}$ and $X_{L+1,N}$ the $P \times N$ matrices defined by

$$X_{1,N} = (x_1, x_2, \ldots, x_N),\ X_{L+1,N} = (x_{L+1}, x_{L+2}, \ldots, x_{N+L}) \tag{4.5}$$

and by $N_{f,N}$ and $N_{p,N}$ the $KL \times N$ matrices defined as the analogues of $Y_{f,N}$ and $Y_{p,N}$ obtained by replacing M–dimensional vectors $(y_n)_{n=1,\ldots,N+2L-1}$ by K–dimensional vectors $(\omega_n)_{n=1,\ldots,N+2L-1}$. Matrices $U_{f,N}$ and $U_{p,N}$ are defined in the same way from $(u_n)_{n=1,\ldots,N+2L-1}$. It is easy to check that

$$U_{p,N} = \mathcal{O}_N\, X_1 + \mathcal{H}_N\, N_{p,N},\ U_{f,N} = \mathcal{O}_N\, X_{L+1,N} + \mathcal{H}_N\, N_{f,N} \tag{4.6}$$

As P, K, L remain fixed, matrix

$$\frac{1}{N} \begin{pmatrix} X_{1,N} \\ N_{p,N} \end{pmatrix} \begin{pmatrix} X^*_{1,N} & N^*_{p,N} \end{pmatrix}$$

converges almost surely towards the covariance matrix of vector $\begin{pmatrix} x_n \\ \omega^L_n \end{pmatrix}$, i.e. matrix

$$\begin{pmatrix} R_x & 0 \\ 0 & I_{KL} \end{pmatrix}$$

As the rank of this matrix is obviously $P + KL$, the same property holds for $\begin{pmatrix} X_{1,N} \\ N_{p,N} \end{pmatrix}$ for N large enough. Moreover, (4.1) implies that

$$\sup_N \|(\mathcal{O}_N, \mathcal{H}_N)\| < +\infty \tag{4.7}$$

from which we deduce that

$$\|R^L_{u,N} - \frac{U_{p,N} U^*_{p,N}}{N}\| \to 0 \tag{4.8}$$

It holds similarly that

$$\|R^L_{u,N} - \frac{U_{f,N} U^*_{f,N}}{N}\| \to 0 \tag{4.9}$$

It is thus clear that the column space of matrices $U_{p,N}$ and $U_{f,N}$ both coincide with the r–dimensional column space of $(\mathcal{O}_N, \mathcal{H}_N)$ for N large enough. We introduce the singular value decompositions of matrices $\frac{U_{p,N}}{\sqrt{N}}$ and $\frac{U_{f,N}}{\sqrt{N}}$:

$$\frac{U_{p,N}}{\sqrt{N}} = \Theta_{p,N}\, \Delta_{p,N} \tilde{\Theta}^*_{p,N},\ \frac{U_{f,N}}{\sqrt{N}} = \Theta_{f,N}\, \Delta_{f,N} \tilde{\Theta}^*_{f,N} \tag{4.10}$$

where $\Theta_{i,N}, \Delta_{i,N}, \tilde{\Theta}_{i,N}$ are $ML \times r$, $r \times r$, $N \times r$ matrices that of course depend on N for $i = p, f$. (4.8) and (4.9) imply that $\|\Theta_{i,N}\Theta^*_{i,N} - \Theta_N \Theta^*_N\| \to 0$ for $i = p, f$ and similarly, that $\|\Delta_{i,N} - \Delta_N\| \to 0$ for $i = p, f$.

We also remark that

$$\frac{1}{N}\begin{pmatrix} X_{L+1,N} \\ N_{f,N} \end{pmatrix} (X_{1,N}^* N_{p,N}^*) \rightarrow \mathbb{E}\left[\begin{pmatrix} x_{n+L+1} \\ \omega_{n+L}^L \end{pmatrix} (x_n^*, \omega_n^{L*})\right] = \begin{bmatrix} \mathbb{E}\left(x_{n+L}(x_n^*, \omega_n^{L*})\right) \\ 0 \end{bmatrix}$$

Therefore, we obtain that

$$\left\| \frac{1}{N} (\mathcal{O}_N, \mathcal{H}_N) \begin{pmatrix} X_{L+1,N} \\ N_{f,N} \end{pmatrix} (X_{1,N}^* N_{p,N}^*) \begin{pmatrix} \mathcal{O}_N^* \\ \mathcal{H}_N^* \end{pmatrix} - (\mathcal{O}_N, \mathcal{H}_N) \begin{pmatrix} \mathbb{E}\left(x_{n+L}\, u_n^{L*}\right) \\ 0 \end{pmatrix} \right\| \rightarrow 0$$

because $u_n^{L*} = (x_n^*, \omega_n^{L*})\mathcal{O}_N^*$ (see (4.3)). It is easily seen that matrix $\mathbb{E}\left(x_{n+L} u_n^{*L}\right)$ coincides with $\mathcal{C}_N = (A^{L-1}G, \ldots, G)$. Notice in particular that matrix G coincides with $\mathbb{E}(x_{n+1} u_n^*)$. Moroever, as $R_{f|p,N}^L = \mathbb{E}(u_{n+L}^L u_n^{*L})$ is equal to $\mathcal{O}_N \mathcal{C}_N$, it holds that

$$\left\| \frac{U_{f,N} U_{p,N}^*}{N} - R_{f|p,N}^L \right\| \rightarrow 0$$

Hence, Rank $\left(\frac{U_{f,N} U_{p,N}^*}{N}\right) = P$ for each N large enough. As $\frac{U_{f,N} U_{p,N}^*}{N}$ coincides with $\Theta_{f,N} \Delta_{f,N} \tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N} \Delta_{p,N} \Theta_{p,N}^*$, we obtain that Rank $\left(\Delta_{f,N} \tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N} \Delta_{p,N}\right) = P$, Rank $\left(\Delta_N \tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N} \Delta_N\right) = P$ and that Rank $\left(\tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N}\right) = P$ for each N large enough. In the following, we denote by Γ_N the rank P $r \times r$ matrix defined by

$$\Gamma_N = \Delta_N \tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N} \Delta_N \tag{4.11}$$

It is clear that

$$\| R_{f|p,N}^L - \Theta_N \Gamma_N \Theta_N^* \| \rightarrow 0 \tag{4.12}$$

If we consider the singular value decomposition

$$\Gamma_N = \Upsilon_N \Xi_N \tilde{\Upsilon}_N^* \tag{4.13}$$

of matrix Γ_N, then, (4.12) implies that the P non zero singular values of $R_{f|p,N}^L$ have the same asymptotic behaviour than the P non zero singular values $(\chi_{k,N})_{k=1,\ldots,P}$ of Γ_N.

4.2 New assumptions and their consequences.

In order to simplify the notations, we denote by $\Sigma_{i,N}$ and $W_{i,N}$ matrices $\Sigma_{i,N} = \frac{Y_{i,N}}{\sqrt{N}}$ and $W_{i,N} = \frac{V_{i,N}}{\sqrt{N}}$ for $i = p, f$. It is easy to check that

$$\Sigma_f \Sigma_p^* = W_f W_p^* + (\Theta_f, W_f \tilde{\Theta}_p \Delta_p) \begin{pmatrix} \Delta_f \tilde{\Theta}_f^* \tilde{\Theta}_p \Delta_p & I_r \\ I_r & 0 \end{pmatrix} \begin{pmatrix} \Theta_p^* \\ \Delta_f \tilde{\Theta}_f^* W_p^* \end{pmatrix} \tag{4.14}$$

We denote by $\mathcal{A}$ and $\mathcal{B}$ the matrices defined by

$$\mathcal{A} = \left(\Theta_f, W_f \tilde{\Theta}_p \Delta_p\right) \tag{4.15}$$

and

$$\mathcal{B} = \left(\Theta_p, W_p \tilde{\Theta}_f \Delta_f\right) \begin{pmatrix} \Delta_p \tilde{\Theta}_p^* \tilde{\Theta}_f \Delta_f & I_r \\ I_r & 0 \end{pmatrix} \tag{4.16}$$

It is easy to check that

$$\begin{pmatrix} -z\, I & \Sigma_f \Sigma_p^* \\ \Sigma_p \Sigma_f^* & -z\, I \end{pmatrix} = \begin{pmatrix} -z\, I & W_f W_p^* \\ W_p W_f^* & -z\, I \end{pmatrix} + \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{pmatrix} \begin{pmatrix} 0 & I_{2r} \\ I_{2r} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}^* & 0 \\ 0 & \mathcal{B}^* \end{pmatrix} \tag{4.17}$$

We denote by $\mathbf{Q}_W(z)$ the resolvent of matrix $\begin{pmatrix} -z\,I & W_f W_p^* \\ W_p W_f^* & -z\,I \end{pmatrix}$. Consider a positive real number y such that y is not eigenvalue of $\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}$ for each N large enough (some conditions on such an eigenvalue will be precised below). For $z = y$, the left handside of (4.17) can also be written as

$$\begin{pmatrix} -y\,I & \Sigma_f \Sigma_p^* \\ \Sigma_p \Sigma_f^* & -y\,I \end{pmatrix} = \begin{pmatrix} -y\,I & W_f W_p^* \\ W_p W_f^* & -y\,I \end{pmatrix} \left(I_{2ML} + \mathbf{Q}_W(y) \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{pmatrix} \begin{pmatrix} 0 & I_{2r} \\ I_{2r} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}^* & 0 \\ 0 & \mathcal{B}^* \end{pmatrix} \right) \tag{4.18}$$

Therefore, y is eigenvalue of $\begin{pmatrix} 0 & \Sigma_f \Sigma_p^* \\ \Sigma_p \Sigma_f^* & 0 \end{pmatrix}$ if and only the determinant of the second term of the right handside of (4.18) vanishes. Using the identity $\det(I + EF) = \det(I + FE)$, we obtain that y is an eigenvalue of $\begin{pmatrix} 0 & \Sigma_f \Sigma_p^* \\ \Sigma_p \Sigma_f^* & 0 \end{pmatrix}$ if and only

$$\det \left(I_{4r} + \begin{pmatrix} \mathcal{A}^* & 0 \\ 0 & \mathcal{B}^* \end{pmatrix} \mathbf{Q}_W(y) \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{pmatrix} \begin{pmatrix} 0 & I_{2r} \\ I_{2r} & 0 \end{pmatrix} \right) = 0 \tag{4.19}$$

or equivalently if

$$\det \left(I_{4r} + F_N(y) \right) = 0 \tag{4.20}$$

where $F_N(z)$ is the $4r \times 4r$ matrix valued function given by

$$F_N(z) = \begin{pmatrix} \mathcal{A}^* \mathbf{Q}_{W,pf}(z) \mathcal{B} & \mathcal{A}^* \mathbf{Q}_{W,pp}(z) \mathcal{A} \\ \mathcal{B}^* \mathbf{Q}_{W,ff}(z) \mathcal{B} & \mathcal{B}^* \mathbf{Q}_{W,fp}(z) \mathcal{A} \end{pmatrix} \tag{4.21}$$

In order to study the asymptotic behaviour of the zeros of Eq. (4.20), it appears necessary to formulate assumptions that allow to precise under which conditions y is not an eigenvalue of $\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}$ for each N large enough, and that insure that matrix $F_N(y)$ has a limit when $N \to +\infty$. Some of these assumptions are mainly purely technical in that they essentially allow to establish well founded mathematical results, but, in practice, we believe that they are not very important. We need to distinguish 3 kinds of extra-assumptions.

— Assumptions on the asymptotic behaviour of the eigenvalue distribution of matrix R_N.

Assumption A-4: *If* $\omega_N = \frac{1}{M} \sum_{k=1}^{M} \delta_{\lambda_{k,N}}$ *is the eigenvalue distribution of matrix* R_N*, it is assumed that*

$$\lim_{N \to +\infty} \lambda_{1,N} = \lambda_{+,*} \quad \lim_{N \to +\infty} \lambda_{M,N} = \lambda_{-,*} \tag{4.22}$$

We note that $\lambda_{-,*} \geq a > 0$ *and* $\lambda_{+,*} \leq b$ *where* a *and* b *are defined by (2.2). Moreover, sequence* $(\omega_N)_{N \geq 1}$ *is assumed to converge weakly towards a probability measure* ω_**, which, necessarily, is carried by* $[\lambda_{-,*}, \lambda_{+,*}]$

Assumption A-5: *It is assumed that for each* N *large enough, eigenvalues* $(\lambda_{k,N})_{k=1,\ldots,M}$ *satisfy condition (3.147), so that support* $\mathcal{S}_N$ *of* μ_N *is equal to* $\mathcal{S}_N = [0, x_{+,N}]$*. Moreover, we add the following condition : for each* N *large enough,*

$$\lambda_{1,N} - \lambda_{k,N} \leq \kappa \, \frac{k-1}{M} \tag{4.23}$$

for some nice constant κ*.*

— Assumptions on the asymptotic behaviour of matrices depending of the useful signal.

Assumption A-6: $r \times r$ *matrices* Δ_N *and* $\Gamma_N = \Delta_N \tilde{\Theta}_{f,N}^* \tilde{\Theta}_{p,N} \Delta_N$ *converge towards matrices* Δ_* *and* Γ_* *respectively. It is moreover assumed that* $\Delta_* > 0$*.*

— Assumptions on the asymptotic behaviour of matrices depending both of the useful signal and the noise.

Assumption A-7: *We denote by* $(f_{k,N})_{k=1,\ldots,M}$ *the eigenvectors of matrix* R_N*, and consider the* $M \times M$ *matrix-valued function positive measure* ω_N^R *defined by*

$$\omega_N^R = \sum_{k=1}^{M} \delta_{\lambda_{k,N}} \, f_{k,N} f_{k,N}^*$$

We introduce the $r \times r$ *matrix-valued measure* γ_N *defined by*

$$d\gamma_N(\lambda) = \Theta_N^* \left(I_L \otimes d\omega_N^R(\lambda) \right) \Theta_N \tag{4.24}$$

Then it is assumed that the sequence $(\gamma_N)_{N \geq 1}$ *converges weakly towards a certain measure* γ_**.*
We have first to establish consequences of Assumption 4 and Assumption 5. The following result holds.

Proposition 4.1. — *We denote by* t_{ω_*} *the Stieltjès transform of limit distribution* ω_**. Then,*

$$\lim_{w \to \lambda_{+,*}, w > \lambda_{+,*}} t_{\omega_*}(w) = -\infty \tag{4.25}$$

- *The sequence* $(w_{+,N})_{N \geq 1}$ *converges towards a finite limit* $w_{+,*}$ *which verifies* $w_{+,*} > \lambda_{+,*}$*.*
- *The sequence* $(x_{+,N})_{N \geq 1}$ *defined by* $x_{+,N} = \phi_N(w_{+,N})$ *converges towards a finite limit* $x_{+,*}$*.*
- *If* $\phi_*(w)$ *is the function defined on* $\mathbb{C} - [\lambda_{-,*}, \lambda_{+,*}]$ *by*

$$\phi_*(w) = (c_* w)^2 \left(\int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{w - \lambda} \right)^2 + c_* w^2 \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{w - \lambda} \tag{4.26}$$

  it holds that

$$x_{+,*} = \phi_*(w_{+,*}) \tag{4.27}$$

- *The sequence* $(\mu_N)_{N \geq 1}$ *converges weakly towards a probability measure* μ_**. The support* $\mathcal{S}_*$ *of* μ_* *is included into* $[0, x_{+,*}]$*, and the Stieltjès transform* $t_*(z)$ *of* μ_* *verifies the equation*

$$t_*(z) = \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda}{-z(1 + \frac{c_* t_*(z) \, \lambda}{1 + z(c_* t_*(z))^2})} \, d\mu_*(\lambda) \tag{4.28}$$

  for each $z \in \mathbb{C} - [0, x_{+,*}]$*.*
- *Moreover, if* $w_*(z)$ *is the function defined on* $\mathbb{C} - [0, x_{+,*}]$ *by*

$$w_*(z) = c_* z t_*(z) - \frac{1}{c_* t_*(z)} \tag{4.29}$$

  then, w_* *is holomorphic on* $\mathbb{C} - [0, x_{+,*}]$ *and verifies*

$$\phi_*(w_*(z)) = z \tag{4.30}$$

  for each $z \in \mathbb{C} - [0, x_{+,*}]$
- $$\lim_{x \to x_{+,*}, x > x_{+,*}} t_*(x) \text{ exists, is finite, is still denoted } t_*(x_{+,*}) \text{, and Eq. (4.28) holds for } z = x_{+,*} \tag{4.31}$$

  Moreover, we have

$$w_{+,*} = w_*(x_{+,*}) \tag{4.32}$$

Proof. We first establish (4.25). We have to prove that for each $A > 0$, it exists $\eta > 0$ such that

$$-t_{\omega_*}(w) = \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{d\omega_*(\lambda)}{w - \lambda} > A$$

whenever $w > \lambda_{+,*}$ and $w - \lambda_{+,*} < \eta$. For this, we consider $w > \lambda_{+,*}$, a condition that implies that $w - \lambda_{1,N} > 0$ is bounded away from 0 for $N > N_0(w)$ large enough. We remark that

$$\frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{k,N}} = \frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{+,*}+\lambda_{+,*}-\lambda_{1,N}+\lambda_{1,N}-\lambda_{k,N}}$$

For each $\delta > 0$, (4.22) implies that it exists $N_1(\delta)$ such that $|\lambda_{+,*} - \lambda_{1,N}| < \delta$ for each $N \geq N_1(\delta)$. Morever, (4.23) leads to $\lambda_{1,N} - \lambda_{k,N} \leq \kappa\left(\frac{k-1}{M}\right)$. Therefore, we obtain that

$$\frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{k,N}} \geq \frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{+,*}+\delta+\kappa\left(\frac{k-1}{M}\right)} \geq \int_0^1 \frac{1}{w-\lambda_{+,*}+\delta+\kappa\, u}\, du$$

Therefore, for each $B > 0$, it exists η such that $w - \lambda_{+,*} < \eta$ and $\delta < \eta$ imply that

$$\int_0^1 \frac{1}{w-\lambda_{+,*}+\delta+\kappa u}\, du > B$$

For these choices of w and δ, it holds that

$$\frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{k,N}} > B$$

for each $N > Max(N_1(\delta), N_0(w))$. The weak convergence of ω_N towards ω_* implies that for each $w > \lambda_{+,*}$ such that $w - \lambda_{+,*} < \eta$, and for each $\gamma > 0$, it exists an integer $N_2(w, \gamma)$ such that

$$\int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{d\omega_*(\lambda)}{w-\lambda} > \frac{1}{M}\sum_{k=1}^{M} \frac{1}{w-\lambda_{k,N}} - \gamma > B - \gamma$$

for each $N \geq Max(N_2(w,\gamma), N_1(\delta), N_0(w))$. Choosing $B = A + \gamma$, we have shown that $-t_{\omega_*}(w) > A$ as soon as $w - \lambda_{+,*} < \eta$.

In order to establish that $(w_{+,N})_{N\geq 1}$ converges towards a finite limit $w_{+,*}$, we first recall that Lemma 3.14 implies that sequences $(w_{+,N})_{N\geq 1}$ and $(x_{+,N})_{N\geq 1}$ are bounded. We put $w_{+,*} = \liminf w_{+,N}$ and $w_+^* = \limsup w_{+,N}$, and establish that $w_{+,*} = w_+^*$. For this, we first prove that $w_{+,*} > \lambda_{+,*}$. We first remark that $w_{+,N} > \lambda_{1,N}$ for each N. Therefore, it holds that $w_{+,*} \geq \lambda_{+,*}$. We thus assume that the equality holds, and show a contradiction by using (4.25). We consider a subsequence $(w_{+,k_N})_{N\geq 1}$ extracted from $(w_{+,N})_{N\geq 1}$ and converging towards $w_{+,*}$, assumed to be equal to $\lambda_{+,*}$. We denote by $M(k_N)$ the value of the dimension of the observations corresponding to the number of observations k_N. $x_{+,k_N} = \phi_{k_N}(w_{+,k_N})$ is given by

$$x_{+,k_N} = (c_{k_N} w_{+,k_N})^2 \left(\int \frac{\lambda\, d\omega_{k_N}(\lambda)}{w_{+,k_N}-\lambda}\right)^2 + c_{k_N} w_{+,k_N}^2 \int \frac{\lambda\, d\omega_{k_N}(\lambda)}{w_{+,k_N}-\lambda}$$

Therefore, it holds that

$$x_{+,k_N} > (c_{k_N} w_{+,k_N})^2 \left(\int \frac{\lambda\, d\omega_{k_N}(\lambda)}{w_{+,k_N}-\lambda}\right)^2 = (c_{k_N} w_{+,k_N})^2 \left(\frac{1}{M(k_N)}\sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{w_{+,k_N}-\lambda_{i,k_N}}\right)^2 \tag{4.33}$$

For each $k = 1, \ldots, M(k_N)$, we express $w_{+,k_N} - \lambda_{i,k_N}$ as

$$w_{+,k_N} - \lambda_{i,k_N} = w_{+,k_N} - \lambda_{+,*} + \lambda_{+,*} - \lambda_{i,k_N}$$

As it is assumed that $w_{+,*} = \lambda_{+,*}$, for each $\delta > 0$, it exists $N_0(\delta)$ for which $w_{+,k_N} - \lambda_{+,*} < \delta$ for each $N > N_0(\delta)$. Therefore, we obtain that

$$\frac{1}{M(k_N)}\sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{w_{+,k_N}-\lambda_{i,k_N}} > \frac{1}{M(k_N)}\sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{\delta+\lambda_{+,*}-\lambda_{i,k_N}}$$

Assumption 4 implies that

$$\lim_{N \to +\infty} \frac{1}{M(k_N)} \sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{\delta + \lambda_{+,*} - \lambda_{i,k_N}} = \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{\delta + \lambda_{+,*} - \lambda}$$

Therefore, for each $\gamma > 0$, it holds that

$$\frac{1}{M(k_N)} \sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{\delta + \lambda_{+,*} - \lambda_{i,k_N}} > \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{\delta + \lambda_{+,*} - \lambda} - \gamma$$

for each $N > N_1(\delta, \gamma)$. (4.25) implies that for each $A > 0$, it exists $\eta > 0$ such that

$$\int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{\delta + \lambda_{+,*} - \lambda} - \gamma > A$$

as soon as $\delta < \eta$. For such a choice of δ, we have shown that

$$\frac{1}{M(k_N)} \sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{w_{+,k_N} - \lambda_{i,k_N}} > \frac{1}{M(k_N)} \sum_{i=1}^{M(k_N)} \frac{\lambda_{i,k_N}}{\delta + \lambda_{+,*} - \lambda_{i,k_N}} > A$$

for each $N > \max(N_0(\delta), N_1(\delta, \gamma))$. Using (4.33), this implies that $\lim_{N \to +\infty} x_{+,k_N} = +\infty$, a contradiction because sequence $(x_{+,N})_{N \geq 1}$ is bounded. This establishes that $w_{+,*} > \lambda_{+,*}$.

We consider function $\phi_*(w)$ defined by (4.26). As $\lambda_{1,N}$ and $\lambda_{M,N}$ are assumed to converge towards $\lambda_{+,*}$ and $\lambda_{-,*}$, the Stieltjès transform $t_{\omega_N}(w)$ of ω_N converges uniformly towards $t_{\omega_*}(w)$ on each compact subset of $\mathbb{C} - [\lambda_{-,*}, \lambda_{+,*}]$. This immediately implies that $\phi_N(w)$ and its derivative $\phi_N'(w)$ converge uniformly towards $\phi_*(w)$ and $\phi_*'(w)$ on each compact subset of $\mathbb{C} - [\lambda_{-,*}, \lambda_{+,*}]$. As $w_{+,*} > \lambda_{+,*}$, sequence w_{+,k_N} stays in a compact subset of $\mathbb{C} - [\lambda_{-,*}, \lambda_{+,*}]$, and therefore, it holds that

$$\phi_{k_N}(w_{+,k_N}) - \phi_*(w_{+,k_N}) \to 0, \; \phi_{k_N}'(w_{+,k_N}) - \phi_*'(w_{+,k_N}) \to 0$$

As $\phi_{k_N}'(w_{+,k_N}) = 0$ for each N, we deduce that $\phi_*'(w_{+,k_N}) \to 0$, and therefore that $\phi_*'(w_{+,*}) = 0$. We obtain similarly that $\phi_*'(w_+^*) = 0$ by introducing a subsequence w_{+,l_N} converging towards w_+^*. We claim that $y \to \phi_*(y)$ is srictly increasing on $]w_{+,*}, +\infty[$. To justify this, we remark that ϕ_N is strictly increasing on $[w_{+,N}, +\infty[$. Therefore, if $w_{+,*} < w_1 < w_2$, it holds that $\phi_N(w_1) < \phi_N(w_2)$, and therefore that $\phi_*(w_1) \leq \phi_*(w_2)$. The equality is impossible because it would imply that $\phi_*'(y) = 0$ for $y \in [w_1, w_2]$, a contradiction because ϕ_* is holomorphic in a neighbourhood of $[w_1, w_2]$. Similarly, ϕ_* is strictly decreasing on $]\lambda_{+,*}, w_{+,*}[$ because ϕ_N is strictly decreasing on $]\lambda_{+,*}, w_{+,N}[$ for each N. If $w_{+,*} < w_+^*$, ϕ_* would have to be strictly decreasing on $]\lambda_{+,*}, w_+^*[$ and strictly increasing on $]w_+^*, +\infty[$ for the same reasons, a contradiction. Therefore, $w_{+,*} = w_+^*$, and $\lim_{N \to +\infty} w_{+,N} = w_{+,*}$. As $\phi_N(w_{+,N}) - \phi_*(w_{+,N}) \to 0$ and that $\phi_*(w_{+,N}) \to \phi_*(w_{+,*})$, we evantually get that $x_{+,N} = \phi_N(w_{+,N})$ converges towards $x_{+,*} = \phi_*(w_{+,*})$. This establishes (4.27).

Proposition 3.2 can be immediately generalized to the case where the eigenvalue distribution $\omega_N = \frac{1}{M} \sum_{k=1}^{M} \delta_{\lambda_{k,N}}$ is replaced by its limit ω_*. Therefore, for each $z \in \mathbb{C}^+$, equation (4.28) has a unique solution $t_*(z)$ such that $t_*(z)$ and $z t_*(z)$ belong to $\mathbb{C}^+$. Function $z \to t_*(z)$ is an element of $\mathcal{S}(\mathbb{R}^+)$, and thus coincides with the Stieltjès transform of a finite positive measure μ_* carried by $\mathbb{R}^+$. Moroever, $\mu_*(\mathbb{R}^+) = \int_{\lambda_{-,*}}^{\lambda_{+,*}} \lambda \, d\omega_*(\lambda) = \lim_{N \to +\infty} \frac{1}{M} \mathrm{Tr}(R_N) = \lim_{N \to +\infty} \mu_N(\mathbb{R}^+)$ and

$$\int_{\mathbb{R}^+} \lambda \, d\mu_*(\lambda) = c_* \int_{\lambda_{-,*}}^{\lambda_{+,*}} \lambda \, d\omega_*(\lambda) \int_{\lambda_{-,*}}^{\lambda_{+,*}} \lambda^2 \, d\omega_*(\lambda) < +\infty \tag{4.34}$$

In order to establish that μ_N converges weakly towards μ_*, it is sufficient to establish that for each $z \in \mathbb{C}^+$, $\lim_{N \to +\infty} t_N(z) = t_*(z)$. The proof is standard, and thus omitted. We just mention that, as in the proof of Proposition 3.2, we need to control $\frac{1}{|1 - z(c_* t_*(z))^2|}$. For this, as in the above proof, we use the observation that

$$\frac{1}{|1 - z(c_* t_*(z))^2|} \leq \frac{1}{|z| \, (c_* \, \mathrm{Im}(t_*(z)))^2}$$

and take benefit of (4.34) to establish, as in the course of the proof of Lemma 3.4, that

$$\mathrm{Im}(t_*(z)) \geq \kappa \, \frac{\mathrm{Im}(z)}{\beta^2 + |z|^2}$$

for some nice constants β and κ. As $x_{+,N}$ converges towards $x_{+,*}$, it holds that $\mu_N([x_{+,*}+\epsilon, +\infty[) = 0$ for each N large enough. As $\mu_N \to \mu_*$, this implies that $\mu_*([x_{+,*}+\epsilon, +\infty[) = 0$ for each $\epsilon > 0$, i.e. $\mu_*(]x_{+,*}, +\infty[) = 0$. Therefore, the support $\mathcal{S}_*$ of μ_* is included into $[0, x_{+,*}]$. Function $t_*(z)$ is thus holomorphic in $\mathbb{C} - [0, x_{+,*}]$. We now justify that (4.28) is still valid if $z = x \in \mathbb{R}^{-*} \cup]x_{+,*}, +\infty[$. For this, it is sufficient to prove that the right handside of (4.28) is holomorphic on $\mathbb{C} - [0, x_{+,*}]$. We consider function $w_*(z)$ defined by (4.29). As t_* is holomorphic in $\mathbb{C} - [0, x_{+,*}]$, so is w_*. Moreover, $w_*(x)$ is of course real if $x > x_{+,*}$ and $x < 0$. Expressing the right handside of (4.28) in terms of $w_*(z)$, we obtain that (4.28) can be written as

$$t_*(z) = \frac{w_*(z)}{z} \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{\lambda \, d\omega_*(\lambda)}{\lambda - w_*(z)} \tag{4.35}$$

If $x < 0$, $w_*(x) < 0$ because $t_*(x) > 0$. Therefore, the right handside of (4.28) is analytic on $\mathbb{C} - \mathbb{R}^+$. In order to show that it is analytic on $\mathbb{C} - [0, x_{+,*}]$, it is sufficient to establish that if $x > x_{+,*}$, then $w_*(x) > \lambda_{+,*}$. We remark that $x > x_{+,N}$ for N large enough. As $t_N(x) \to t_*(x)$ when $N \to +\infty$, we get that $w_N(x) \to w_*(x)$. As $x > x_{+,N}$, it holds that $w_N(x) > w_N(x_{+,N}) = w_{+,N}$. Therefore, $w_*(x) = \lim_{N \to +\infty} w_N(x) \geq \lim_{N \to +\infty} w_{+,N} = w_{+,*}$. As $w_{+,*} > \lambda_{+,*}$, we conclude that $w_*(x) > \lambda_{+,*}$ as expected. Therefore, (4.28) holds on $\mathbb{C} - [0, x_{+,*}]$. We notice that this implies that $\phi_*(w_*(z)) = z$ for each $z \in \mathbb{C} - [0, x_{+,*}]$, i.e. that (4.30) holds.

As function $x \to t_*(x)$ is increasing on $]x_{+,*}, +\infty[$, $\lim_{x \to x_{+,*}, x > x_{+,*}} t_*(x)$ exists. In order to check that it is finite, we remark that $w_*(x) \geq w_{+,*} > \lambda_{+,*}$ for $x > x_{+,*}$. Hence, the right handside of (4.35) remains finite on $]x_{+,*}, +\infty[$. This implies that $\lim_{x \to x_{+,*}, x > x_{+,*}} t_*(x)$ is finite. We deduce from (4.29) that $w_*(x)$ also converges towards a finite limit when $x \to x_{+,*}$. This limit is still denoted by $w_*(x_{+,*})$. We establish that $w_*(x_{+,*}) = w_{+,*}$. For this, we remark that $\phi_*(w_*(x)) = x$ for $x > x_{+,*}$. As $w_*(x) \geq w_{+,*} > \lambda_{+,*}$, $\phi_*(w_*(x))$ converges towards $\phi_*(w_*(x_{+,*})) = x_{+,*}$ when $x \to x_{+,*}$. Moreover, $x_{+,*} = \phi_*(w_*(x_{+,*})) \geq \phi_*(w_{+,*})$. (4.27) thus implies that $\phi_*(w_*(x_{+,*})) = \phi_*(w_{+,*})$ As ϕ_* is strictly increasing on $]w_{+,*}, +\infty[$, this implies that $w_*(x_{+,*}) = w_{+,*}$. We also remark that taking the limit in (4.35) when $x \to x_{+,*}$ leads to the conclusion that (4.35), and therefore (4.28), also hold for $x = x_{+,*}$. This completes the proof of Proposition 4.1.

We recall that ν_N^T is the $M \times M$ matrix-valued positive measure associated to matrix-valued Stieltjès transform $T_N(z)$, and introduce for each N the $r \times r$ matrix-valued measure β_N defined by

$$d\beta_N(\lambda) = \Theta_N^* \left(I_L \otimes d\nu_N^T(\lambda) \right) \Theta_N \tag{4.36}$$

Then, the following result is a consequence of Assumption 7.

Proposition 4.2. *The sequence of measures $(\beta_N)_{N \geq 1}$ converges weakly towards a measure β_* whose support is included into $[0, x_{+,*}]$. The Stieltjès transform $T_{\beta_*}(z)$ of β_* is given by*

$$T_{\beta_*}(z) = \frac{w_*(z)}{z} \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{d\gamma_*(\lambda)}{\lambda - w_*(z)} \tag{4.37}$$

for each $z \in \mathbb{C} - [0, x_{+,*}]$. Moreover, if $T_{\beta_N}(z)$ represents the Stieltjès transform of β_N, then, it holds that

$$T_{\beta_*}(x_{+,*}) = \lim_{x \to x_{+,*}, x > x_{+,*}} T_{\beta_*}(x) = \lim_{N \to +\infty} T_{\beta_N}(x_{+,N}) = \frac{w_{+,*}}{x_{+,*}} \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{d\gamma_*(\lambda)}{\lambda - w_{+,*}} \tag{4.38}$$

Proof. We first notice that $\gamma_N([0, \lambda_{1,N}]) = I_r$ and $\gamma_N(]\lambda_{1,N}, +\infty[) = 0$, and that, for each $\epsilon > 0$, $\gamma_N([0, \lambda_{+,*} + \epsilon]) = I_r$ and $\gamma_N(]\lambda_{+,*}+\epsilon, +\infty[) = 0$ for each N large enough. As $\gamma_N \to \gamma_*$, we obtain that $\gamma_*([0, \lambda_{+,*}+\epsilon]) = I_r$ and $\gamma_*(]\lambda_{+,*} + \epsilon, +\infty[) = 0$ for each $\epsilon > 0$. This implies that $\gamma_*([0, \lambda_{+,*}]) = I_r$ and that $\gamma_*(]\lambda_{+,*}, +\infty[) = 0$.

As $\gamma_N \rightarrow \gamma_*$, the Stieltjès transform $T_{\gamma_N}(w)$ of γ_N converges towards the Stieltjès transform $T_{\gamma_*}(w)$ of γ_* for each $w \in \mathbb{C} - [0, \lambda_{+,*}]$. In other words, for each $z \in \mathbb{C} - [0, \lambda_{+,*}]$, it holds that

$$\int_0^{\lambda_{1,N}} \frac{d\gamma_N(\lambda)}{\lambda - w} = \Theta_N^* \left(I_L \otimes (R_N - wI)^{-1}\right) \Theta_N \rightarrow \int_0^{\lambda_{+,*}} \frac{d\gamma_*(\lambda)}{\lambda - w} \tag{4.39}$$

The convergence is moreover uniform on each compact subset $\mathbb{C} - [0, \lambda_{+,*}]$. For each $z \in \mathbb{C} - [0, x_{+,N}]$, matrix $T_N(z)$ can be written as

$$T_N(z) = \frac{w_N(z)}{z}(R_N - w_N(z)I)^{-1}$$

This relation also holds for each $z \in \mathbb{C} - [0, x_{+,*}]$ if N is large enough. Therefore, $T_{\beta_N}(z)$ is given by

$$T_{\beta_N}(z) = \frac{w_N(z)}{z}\Theta_N^* \left(I_L \otimes (R_N - w_N(z)I)^{-1}\right) \Theta_N = \frac{w_N(z)}{z} \int_0^{\lambda_{1,N}} \frac{d\gamma_N(\lambda)}{\lambda - w_N(z)}$$

We now prove that for each $x > x_{+,*}$, then

$$T_{\beta_N}(x) \rightarrow \frac{w_*(x)}{x} \int_{\lambda_{-,*}}^{\lambda_{+,*}} \frac{d\gamma_*(\lambda)}{\lambda - w_*(x)} \tag{4.40}$$

For this, we first notice that $w_N(x) \rightarrow w_*(x)$, and remark that

$$\int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{\lambda - w_N(x)} - \int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{\lambda - w_*(x)} = (w_N(x) - w_*(x)) \int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{(\lambda - w_N(x))(\lambda - w_*(x))}$$

As $w_N(x) \rightarrow w_*(x)$, for N large enough, $w_N(x) - \lambda_{+,*} > \frac{1}{2}(w_*(x) - \lambda_{+,*}) > 0$. Therefore, $\int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{(\lambda - w_N(x))(\lambda - w_*(x))}$ is upper bounded, and

$$\int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{\lambda - w_N(x)} - \int_0^{\lambda_{+,*}} \frac{d\gamma_N(\lambda)}{\lambda - w_*(x)} \rightarrow 0$$

when $N \rightarrow +\infty$. Hence, (4.39) implies that

$$T_{\beta_N}(x) - \frac{w_*(x)}{x} T_{\gamma_N}(w_*(x)) \rightarrow 0$$

As $\gamma_N \rightarrow \gamma_*$, $T_{\gamma_N}(w_*(x))$ converges towards $T_{\gamma_*}(w_*(x))$. Therefore, we have established (4.40). As functions T_{β_N} are Stieltjès transforms, Montel's theorem also implies that (4.40) also holds for each $z \in \mathbb{C} - [0, x_{+,*}]$. Moreover, function $z \rightarrow \frac{w_*(z)}{z} T_{\gamma_N}(w_*(z))$ is the Stieltjès transform of a $r \times r$–valued positive measure β_* carried by $\mathbb{R}^+$. It is moreover easy to check that $\beta_*(\mathbb{R}^+) = I_r$. The convergence of $T_{\beta_N}(z)$ towards $T_{\beta_*}(z)$ thus implies that $(\beta_N)_{N \geq 1}$ converges weakly towards β_*. For each N, β_N is supported by $[0, x_{+,N}]$, and $\beta_N([0, x_{+,N}]) = I_r$. Therefore, for each $\delta > 0$, it holds that $\beta_N([0, x_{+,*} + \delta]) = I_r$ and $\beta_N(]x_{+,*} + \delta, +\infty[) = 0$. Moreover, the support of β_* is included into $[0, x_{+,*}]$ so that $\beta_*([0, x_{+,*}] = I_r$.

We finally establish (4.38). For this, we remark that

$$T_{\beta_N}(x_{+,N}) = \frac{w_{+,N}}{x_{+,N}} \Theta_N^* \left(I_L \otimes (R_N - w_{+,N})^{-1}\right) \Theta_N$$

It is clear that $\frac{w_{+,N}}{x_{+,N}} \rightarrow \frac{w_{+,*}}{x_{+,*}} = \frac{w_*(x_{+,*})}{x_{+,*}}$ and that $\Theta_N^* \left(I_L \otimes (R_N - w_{+,N})^{-1}\right) \Theta_N$ has the same asymptotic behaviour than $\Theta_N^* \left(I_L \otimes (R_N - w_{+,*})^{-1}\right) \Theta_N$, which itself converges towards $T_{\gamma_*}(w_{+,*}) = T_{\gamma_*}(w_*(x_{+,*}))$. We have thus proved that

$$\lim_{N \rightarrow +\infty} T_{\beta_N}(x_{+,N}) = \frac{w_{+,*}}{x_{+,*}} T_{\gamma_*}(w_*(x_{+,*}))$$

Therefore, (4.37) immediately implies (4.38).

We finally conclude this paragraph by characterizing the set of all positive real numbers y that are not eigenvalue of matrix $\begin{pmatrix} 0 & W_f W_p^* \\ W_p W_f^* & 0 \end{pmatrix}$.

Proposition 4.3. *Assume that $y > \sqrt{x_{+,*}}$. Then, for each N large enough, y is not eigenvalue of matrix* $\begin{pmatrix} 0 & W_{f,N}W_{p,N}^* \\ W_{p,N}W_{f,N}^* & 0 \end{pmatrix}$, *and y^2 is not eigenvalue of $\Sigma_f \Sigma_p^* \Sigma_p \Sigma_f^*$.*

Proof. As $y > \sqrt{x_{+,*}}$, it exists N_0 such that $y > \sqrt{x_{+,N}}$ for each $N \geq N_0$. Therefore, y does not belong to $\cup_{N \geq N_0} \mathcal{S}_N$. Theorem 3.4 thus implies that y cannot be one of the eigenvalues of a matrix $\begin{pmatrix} 0 & W_{f,N}W_{p,N}^* \\ W_{p,N}W_{f,N}^* & 0 \end{pmatrix}$ for $N \geq N_0$.

4.3 Asymptotic behaviour of the eigenvalues of $\Sigma_f \Sigma_p^* \Sigma_p \Sigma_f^*$.

In this paragraph, we characterize the possible eigenvalues of $\Sigma_f \Sigma_p^* \Sigma_p \Sigma_f^*$ that escape from the interval $[0, x_{+,*}]$, or equivalently, the positive eigenvalues of $\begin{pmatrix} 0 & \Sigma_{f,N}\Sigma_{p,N}^* \\ \Sigma_{p,N}\Sigma_{f,N}^* & 0 \end{pmatrix}$ that are strictly greater than $\sqrt{x_{+,*}}$. Almost surely, for each N large enough, function $F_N(z)$ defined by (4.21) is holomorphic on $\mathbb{C} - [-\sqrt{x_{+,N}}, \sqrt{x_{+,N}}]$. and on $\mathbb{C} - [-\sqrt{x_{+,*} + \delta}, \sqrt{x_{+,*} + \delta}]$ for each $\delta > 0$.

We first establish that the sequence of analytic functions $(F_N(z))_{N \geq 1}$ almost surely converges uniformly on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards a deterministic function $F_*(z)$ which is analtyic in $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. Using a classical stability result of the zeros of an analytic function (see [4] and [10]), this will imply that the solutions of the equation $\det(I + F_N(y)) = 0$, $y > \sqrt{x_{+,*}}$, will converge towards the solutions of the limit equation $\det(I + F_*(y)) = 0$.

In order to study the asymptotic behaviour of F_N, we first consider the asymptotic behaviour of matrix $\mathcal{A}^* \mathbf{Q}_{W,N}(pf) \mathcal{B}$, which is given by

$$\mathcal{A}^* \mathbf{Q}_{W,N}(pf) \mathcal{B} = \begin{pmatrix} \Theta_f^* \\ \Delta_p \tilde{\Theta}_p^* W_f^* \end{pmatrix} \mathbf{Q}_{W,N}(pf) \begin{pmatrix} \Theta_p, W_p \tilde{\Theta}_f \Delta_f \end{pmatrix} \begin{pmatrix} \Delta_p \tilde{\Theta}_p^* \tilde{\Theta}_f \Delta_f & I_r \\ I_r & 0 \end{pmatrix}$$

In order to study matrix $\mathcal{A}^* \mathbf{Q}_{W,N}(pf) \mathcal{B}$ when $N \to +\infty$, it is necessary to evaluate the asymptotic behaviour of sesquilinear forms of matrices $\mathbf{Q}_{W,N}(pf)$, $W_f^* \mathbf{Q}_{W,N}(pf)$, $\mathbf{Q}_{W,N}(pf) W_p$ and $W_f^* \mathbf{Q}_{W,N}(pf) W_p$. The following result holds.

Lemma 4.1. *For each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ and for each bounded sequences $(a_N, b_N)_{N \geq 1}$ and $\tilde{a}_N, \tilde{b}_N$ of ML–dimensional and N–dimensional deterministic vectors, it holds that*

- $a_N^* \, \mathbf{Q}_{W,N}(pf) \, b_N \to 0$ *almost surely*
- $\tilde{a}_N^* \, W_f^* \, \mathbf{Q}_{W,N}(pf) \, b_N \to 0$ *almost surely*
- $a_N^* \, \mathbf{Q}_{W,N}(pf) \, W_p \, \tilde{b}_N \to 0$ *almost surely*
- $\tilde{a}_N^* \, W_f^* \, \mathbf{Q}_{W,N}(pf) \, W_p \, \tilde{b}_N + \frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2} \tilde{a}_N^* \tilde{b}_N \to 0$ *almost surely.*

Moreover, the convergence is uniform over each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ and it holds that, almost surely

$$\mathcal{A}^* \, \mathbf{Q}_{W,N}(pf) \, \mathcal{B} - \begin{pmatrix} 0 & 0 \\ -\frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2} \Gamma_N^* & 0 \end{pmatrix} \to 0 \tag{4.41}$$

the convergence being uniform on compact subsets of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$.

Sketch of proof. The proof of this result uses ingredients that are very similar to the calculations of Paragraphs 3.4 and 3.5.2. We therefore only provide a sketch of proof. When $z \in \mathbb{C}^+$, the first item follows from $\mathbb{E}(\mathbf{Q}_{W,N}(pf)) = 0$ and the Nash-Poincaré inequality. The convergence for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ follows from the observation that almost surely, for each $\delta > 0$, functions $(a_N^* \, \mathbf{Q}_{W,N}(pf) \, b_N)$ are analytic on $\mathbb{C} - [-\sqrt{x_{+,*} + \delta}, \sqrt{x_{+,*} + \delta}]$ for N large enough. The use of Montel's theorem allows to prove the almost sure convergence towards for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$, as well as the uniformity of the convergence on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. To establish the second and the third item when $z \in \mathbb{C}^+$, it is sufficient to establish that $\mathbb{E}(W_f^* \, \mathbf{Q}_{W,pf}) = 0$, $\mathbb{E}(\mathbf{Q}_{W,pf} \, W_p) = 0$, to use the Nash-Poincaré inequality, and to extend the convergence domain using the Montel's theorem. We note that the sequences of functions defined in item (ii) and (iii) are almost surely bounded on each compact subsets of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ because matrices W_f and W_p are almost surely bounded.

The proof of the last item needs to use the calculations of paragraph 3.4 to establish that

$$\tilde{a}_N^* \, W_f^* \, \mathbf{Q}_{W,N}(pf) \, W_p \, \tilde{b}_N + \frac{(c_N \boldsymbol{\alpha}_N(z))^2}{1 - (c_N \boldsymbol{\alpha}_N(z))^2} \, \tilde{a}_N^* \tilde{b}_N \to 0 \; a.s.$$

for each $z \in \mathbb{C}^+$. It is proved in Paragraph 3.5.2 that $\alpha_N(z) - t_N(z) \to 0$ for each $z \in \mathbb{C}^+$. As $\boldsymbol{\alpha}_N(z) = z\alpha_N(z^2)$ and $\mathbf{t}_N(z) = z t_N(z^2)$, this implies that $\boldsymbol{\alpha}_N(z) - \mathbf{t}_N(z) \to 0$ if $\mathrm{Arg}(z) \in]0, \pi/2[$. This convergence domain can be extended to $\mathbb{C}^+$ using classical arguments based Montel's theorem. From this, we deduce immediately that

$$\frac{(c_N \boldsymbol{\alpha}_N(z))^2}{1 - (c_N \boldsymbol{\alpha}_N(z))^2} - \frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2} \to 0$$

for each $z \in \mathbb{C}^+$, and that, for each $z \in \mathbb{C}^+$,

$$\tilde{a}_N^* \, W_f^* \, \mathbf{Q}_{W,N}(pf) \, W_p \, \tilde{b}_N + \frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2} \, \tilde{a}_N^* \tilde{b}_N \to 0, \; a.s. \tag{4.42}$$

Matrices W_f and W_p are almost surely bounded. Therefore, for each $\delta > 0$, $\tilde{a}_N^* \, W_f^* \, \mathbf{Q}_{W,N}(pf) \, W_p \, \tilde{b}_N$ and $\frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2}$ are analytic on $\mathbb{C} - [-\sqrt{x_{+,*} + \delta}, \sqrt{x_{+,*} + \delta}]$ and bounded on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. Montel's theorem thus implies that (4.42) holds for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. Moreover, the convergence is uniform on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$.

(4.41) is an immediate consequence of the statements of items (i) to (iv) and of the observation that $r \times r$ diagonal matrices $\Delta_{p,N}$ and $\Delta_{f,N}$ (resp. orthogonal $ML \times r$ matrices $\Theta_{f,N}$ and $\Theta_{p,N}$) have the same asymptotic behaviour than matrix Δ_N (resp. matrix Θ_N).

Using the same kind of arguments as in the proof of Lemma 4.1, it is possible to establish the following result.

Proposition 4.4. *For each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$, it holds that*

$$\mathcal{A}^* \, \mathbf{Q}_{W,N}(pp) \, \mathcal{A} - \begin{pmatrix} -\Theta_N^* \left(zI + \frac{c_N \mathbf{t}_N(z)}{1 - (c_N \mathbf{t}_N(z))^2} I_L \otimes R_N \right)^{-1} \Theta_N & 0 \\ 0 & \frac{c_N \mathbf{t}_N(z)}{1 - (c_N \mathbf{t}_N(z))^2} \Delta_N^2 \end{pmatrix} \to 0 \; a.s. \tag{4.43}$$

$$\mathcal{B}^* \, \mathbf{Q}_{W,N}(ff) \, \mathcal{B} - \begin{pmatrix} \Gamma_N^* & I \\ I & 0 \end{pmatrix} \begin{pmatrix} -\Theta_N^* \left(zI + \frac{c_N \mathbf{t}_N(z)}{1 - (c_N \mathbf{t}_N(z))^2} I_L \otimes R_N \right)^{-1} \Theta_N & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} \Gamma_N & I \\ I & 0 \end{pmatrix} \to 0 \; a.s. \tag{4.44}$$

$$\mathcal{B}^* \, \mathbf{Q}_{W,N}(ff) \, \mathcal{A} - \begin{pmatrix} 0 & -\frac{(c_N \mathbf{t}_N(z))^2}{1 - (c_N \mathbf{t}_N(z))^2} \Gamma_N \\ 0 & 0 \end{pmatrix} \to 0 \; a.s. \tag{4.45}$$

The convergence is moreover uniform on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$.

Lemma 4.1 and Proposition 4.4 imply that for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$, almost surely, matrix $F_N(z)$ has the same asymptotic behaviour than the $4r \times 4r$ deterministic matrix $F_{d,N}(z)$ defined by

$$F_{d,N}(z) = \begin{pmatrix} F_{d,N}^{11}(z) & F_{d,N}^{1,2}(z) \\ F_{d,N}^{2,1}(z) & F_{d,N}^{2,2}(z) \end{pmatrix} \tag{4.46}$$

where the $2r \times 2r$ blocks of $F_{d,N}(z)$ are characterized in Lemma 4.1 and in Proposition 4.4. The assumptions formulated in Paragraph 4.2 imply that matrix $F_{d,N}(z)$ converges for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards a limit $F_*(z)$, the convergence being uniform on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. More precisely, $t_N(z)$ converges towards $t_*(z)$ uniformly on each compact subset of $\mathbb{C} - [0, x_{+,*}]$, which implies that $\mathbf{t}_N(z) = zt_N(z^2)$ converges uniformly on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards $\mathbf{t}_*(z) = zt_*(z^2)$. We notice that matrix $-\left(zI + \frac{c_N \mathbf{t}_N(z)}{1-(c_N \mathbf{t}_N(z))^2} I_L \otimes R_N\right)^{-1}$ coincides with matrix $I_L \otimes \mathbf{T}_N(z) = I_L \otimes zT_N(z^2) = z\int_0^{x_{+,N}} \frac{I_L \otimes d\nu_N^T(\lambda)}{\lambda - z^2}$. We denote by $\mathbf{T}_{\beta_N}(z)$ the function defined by $\mathbf{T}_{\beta_N}(z) = zT_{\beta_N}(z^2)$, which can also be written as

$$\Theta_N^*(I_L \otimes \mathbf{T}_N(z))\Theta_N = \mathbf{T}_{\beta_N}(z)$$

Assumption 6 implies that $\mathbf{T}_{\beta_N}(z)$ converges uniformly on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards the $r \times r$ matrix $\mathbf{T}_{\beta_*}(z)$ defined by

$$\mathbf{T}_{\beta_*}(z) = zT_{\beta_*}(z^2) \tag{4.47}$$

where we recall that $T_{\beta_*}(z) = \int_0^{x_{+,*}} \frac{d\beta_*(\lambda)}{\lambda - z}$ is the Stieltjès transform of the positive matrix-valued measure β_*. All this imply that

$$F_{d,N}^{(1,1)}(z) = \begin{pmatrix} 0 & 0 \\ -\frac{(c_N \mathbf{t}_N(z))^2}{1-(c_N \mathbf{t}_N(z))^2}\Gamma_N^* & 0 \end{pmatrix} \rightarrow F_*^{1,1}(z) = \begin{pmatrix} 0 & 0 \\ -\frac{(c_* \mathbf{t}_*(z))^2}{1-(c_* \mathbf{t}_*(z))^2}\Gamma_*^* & 0 \end{pmatrix}$$

$$F_{d,N}^{(1,2)}(z) = \begin{pmatrix} \mathbf{T}_{N,\beta}(z) & 0 \\ 0 & \frac{c_N \mathbf{t}_N(z)}{1-(c_N \mathbf{t}_N(z))^2}\Delta_N^2 \end{pmatrix} \rightarrow F_*^{1,2}(z) = \begin{pmatrix} \mathbf{T}_{\beta_*}(z) & 0 \\ 0 & \frac{c_* \mathbf{t}_*(z)}{1-(c_* \mathbf{t}_*(z))^2}\Delta_*^2 \end{pmatrix}$$

$$F_{d,N}^{2,1}(z) = \begin{pmatrix} \Gamma_N & I \\ I & 0 \end{pmatrix} F_{d,N}^{1,2}(z) \begin{pmatrix} \Gamma_N^* & I \\ I & 0 \end{pmatrix} \rightarrow F_*^{2,1}(z) = \begin{pmatrix} \Gamma_* & I \\ I & 0 \end{pmatrix} F_*^{1,2}(z) \begin{pmatrix} \Gamma_*^* & I \\ I & 0 \end{pmatrix}$$

$$F_{d,N}^{2,2}(z) = \begin{pmatrix} 0 & -\frac{(c_N \mathbf{t}_N(z))^2}{1-(c_N \mathbf{t}_N(z))^2}\Gamma_N \\ 0 & 0 \end{pmatrix} \rightarrow F_*^{1,1}(z) = \begin{pmatrix} 0 & -\frac{(c_* \mathbf{t}_*(z))^2}{1-(c_* \mathbf{t}_*(z))^2}\Gamma_* \\ 0 & 0 \end{pmatrix}$$

where we recall that Γ_* is defined by Assumption 6. The previous results show that $(F_N(z))_{N \geq 1}$ converge uniformly towards $F_*(z)$ over each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. It is thus reasonable to expect that the solutions of the equation $\det(I + F_N(y)) = 0$ satisfying $y > \sqrt{x_{+,*}}$ will converge towards the roots of $\det(I + F_*(y)) = 0$ satisfying $y > \sqrt{x_{+,*}}$. In order to establish this, we use in the following the classical stability argument used in [4], Lemma 6.1 (see also [10]). Before invoking [4], we have first to study the solutions of $\det(I + F_*(y)) = 0$.

For $y > \sqrt{x_{+,*}}$, we now express in a more convenient manner the equation $\det(I + F_*(y)) = 0$. This equation holds if and only

$$\det\left(\begin{pmatrix} I & 0 \\ 0 & \Omega_* \end{pmatrix}(I + F_*(y))\begin{pmatrix} I & 0 \\ 0 & \Omega_*^* \end{pmatrix}\right) = 0 \tag{4.48}$$

where

$$\Omega_* = \begin{pmatrix} \Gamma_* & I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & -\Gamma_* \end{pmatrix}$$

The matrix whose determinant vanishes in (4.48) is equal to

$$\begin{pmatrix} 0 & I & \mathbf{T}_{\beta_*}(z) & 0 \\ I & -\frac{\Gamma_*^*}{(1-(c_* \mathbf{t}_*(z))^2} & 0 & \frac{c_N \mathbf{t}_*(z)}{1-(c_* \mathbf{t}_*(z))^2}\Delta_*^2 \\ \mathbf{T}_{\beta_*}(z) & 0 & 0 & I \\ 0 & \frac{c_* \mathbf{t}_*(z)}{1-(c_* \mathbf{t}_*(z))^2}\Delta_*^2 & I & -\frac{\Gamma_*}{(1-(c_* \mathbf{t}_*(z))^2} \end{pmatrix} \tag{4.49}$$

As the lower diagonal $2r \times 2r$ block of this matrix is invertible, its determinant is 0 if and only the determinant of its Schur complement is 0. After some calculations, we obtain that $\det(I + F_*(y)) = 0$ if and only if $\det(I - G_*(y)) = 0$ where $G_*(z)$ is the $2r \times 2r$ matrix-valued function defined for each $z \in \mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ by

$$G_*(z) = \begin{pmatrix} \frac{c_* \mathbf{t}_*(z)}{1-(c_* \mathbf{t}_*(z))^2}\Delta_*^2 \mathbf{T}_{\beta_*}(z) & \frac{\Gamma_*^*}{(1-(c_* \mathbf{t}_*(z))^2} \\ \frac{\mathbf{T}_{\beta_*}(z)\,\Gamma_* \mathbf{T}_{\beta_*}(z)}{1-(c_* \mathbf{t}_*(z))^2} & \frac{c_* \mathbf{t}_*(z)}{1-(c_* \mathbf{t}_*(z))^2}\mathbf{T}_{\beta_*}(z)\,\Delta_*^2 \end{pmatrix} \tag{4.50}$$

$G_*(z)$ can be factorized as

$$G_*(z) = \begin{pmatrix} I & 0 \\ 0 & \mathbf{T}_{\beta_*}(z) \end{pmatrix} \begin{pmatrix} \frac{c_*\mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 & \frac{\Gamma_*^*}{(1-(c_*\mathbf{t}_*(z))^2} \\ \frac{\Gamma_*}{(1-(c_*\mathbf{t}_*(z))^2} & \frac{c_*\mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 \end{pmatrix} \begin{pmatrix} \mathbf{T}_{\beta_*}(z) & 0 \\ 0 & I \end{pmatrix}$$

For each $y > \sqrt{x_{+,*}}$, matrix $\mathbf{T}_{\beta_*}(y)$ is negative definite, and thus invertible. Therefore, $\det(I - G_*(y)) = 0$ if and only

$$\det\left(\begin{pmatrix} \frac{c_*\mathbf{t}_*(y)}{1-(c_*\mathbf{t}_*(y))^2}\Delta_*^2 & \frac{\Gamma_*^*}{(1-(c_*\mathbf{t}_*(y))^2} \\ \frac{\Gamma_*}{(1-(c_*\mathbf{t}_*(y))^2} & \frac{c_*\mathbf{t}_*(y)}{1-(c_*\mathbf{t}_*(y))^2}\Delta_*^2 \end{pmatrix} - \begin{pmatrix} (\mathbf{T}_{\beta_*}(y))^{-1} & 0 \\ 0 & (\mathbf{T}_{\beta_*}(y))^{-1} \end{pmatrix} \right) = 0 \tag{4.51}$$

In the following, we denote by $H_*(z)$ the $2r \times 2r$ matrix-valued defined on $\mathbb{C} - [\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ by

$$H_*(z) = \begin{pmatrix} \frac{c_*\mathbf{t}_*(z))}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 - (\mathbf{T}_{\beta_*}(z))^{-1} & \frac{\Gamma_*^*}{(1-(c_*\mathbf{t}_*(z))^2} \\ \frac{\Gamma_*}{(1-(c_*\mathbf{t}_*(z))^2} & \frac{c_*\mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 - (\mathbf{T}_{\beta_*}(z))^{-1} \end{pmatrix} \tag{4.52}$$

$H_*(z)$ is of course holomorphic on $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$, and the solutions of $\det(I + F_*(y)) = 0$ coincide with the solutions of

$$\det\left(H_*(y)\right) = 0 \tag{4.53}$$

where $y > \sqrt{x_{+,*}}$. In order to characterize the roots of (4.53), we first establish the following Proposition.

Proposition 4.5. *For each $z \in \mathbb{C}^+$, $\mathrm{Im}(H_*(z)) > 0$, and function $y \to H_*(y)$ is increasing in the sense of the partial order defined on the set of all Hermitian matrices on the interval $[\sqrt{x_{+,*}}, +\infty[$.*

Proof. It is clear that $\mathrm{Im}\left((\mathbf{T}_{\beta_*}(z))^{-1}\right) < 0$ for each $z \in \mathbb{C}^+$. Therefore, in order to establish that $\mathrm{Im}(H_*(z)) > 0$ on $\mathbb{C}^+$, it is sufficient to prove that $\mathrm{Im}(H_{*,1}(z)) > 0$ on $\mathbb{C}^+$ where $H_{*,1}(z)$ is the function defined by

$$H_{*,1}(z) = \begin{pmatrix} \frac{c_*\mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 & \frac{\Gamma_*^*}{(1-(c_*\mathbf{t}_*(z))^2} \\ \frac{\Gamma_*}{(1-(c_*\mathbf{t}_*(z))^2} & \frac{c_*\mathbf{t}_*(z)}{1-(c_*\mathbf{t}_*(z))^2}\Delta_*^2 \end{pmatrix}$$

After some calculations, we obtain that

$$\mathrm{Im}(H_{*,1}(z)) = \frac{1}{|1-(c_*\mathbf{t}_*(z))^2|^2} \begin{pmatrix} \mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\Delta_*^2 & \mathrm{Im}\left((c_*t_*(z))^2\right)\Gamma_*^* \\ \mathrm{Im}\left((c_*t_*(z))^2\right)\Gamma_* & \mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\Delta_*^2 \end{pmatrix}$$

It is clear that $\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\Delta_*^2 > 0$. Therefore, $\mathrm{Im}(H_{*,1}(z)) > 0$ if and only if

$$\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\Delta_*^2 - \frac{\left[\mathrm{Im}\left((c_*t_*(z))^2\right)\right]^2}{\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)}\Gamma_*^*\Delta_*^{-2}\Gamma_* > 0$$

or equivalently, if and only if

$$I - \frac{\left[\mathrm{Im}\left((c_*t_*(z))^2\right)\right]^2}{\left[\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\right]^2}\Delta_*^{-1}\Gamma_*^*\Delta_*^{-2}\Gamma_*\Delta_*^{-1} > 0 \tag{4.54}$$

We first claim that $\Delta_*^{-1}\Gamma_*^*\Delta_*^{-2}\Gamma_*\Delta_*^{-1} \leq I$. To verify this, we notice that for each N, matrix $\Delta_N^{-1}\Gamma_N^*\Delta_N^{-2}\Gamma_N\Delta_N^{-1}$ coincides with $\tilde{\Theta}_{f,N}^*\tilde{\Theta}_{p,N}\tilde{\Theta}_{p,N}^*\tilde{\Theta}_{f,N}$ which is less than I. Therefore,

$$\lim_{N\to+\infty} \Delta_N^{-1}\Gamma_N^*\Delta_N^{-2}\Gamma_N\Delta_N^{-1} = \Delta_*^{-1}\Gamma_*^*\Delta_*^{-2}\Gamma_*\Delta_*^{-1} \leq I$$

$\frac{\left[\mathrm{Im}\left((c_*t_*(z))^2\right)\right]^2}{\left[\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\right]^2}$ is equal to

$$\frac{\left[\mathrm{Im}\left((c_*t_*(z))^2\right)\right]^2}{\left[\mathrm{Im}(c_*t_*(z))(1+|c_*t_*(z)|^2)\right]^2} = \frac{4\left[\mathrm{Re}(c_*t_*(z))\right]^2}{(1+|c_*t_*(z)|^2)^2}$$

For $z \in \mathbb{C}^+$, $\text{Im}(t_*(z)) > 0$. Therefore, it holds that $(\text{Re}(c_* t_*(z)))^2 < |c_* t_*(z)|^2$ and that

$$\frac{\left[\text{Im}\left((c_* t_*(z))^2\right)\right]^2}{\left[\text{Im}(c_* t_*(z))(1 + |c_* t_*(z)|^2)\right]^2} < \frac{4|c_* t_*(z)|^2}{(1 + |c_* t_*(z)|^2)^2} \leq 1$$

This establishes (4.54) and $\text{Im}(H_*(z)) > 0$.

We now prove that $y \to H_*(y)$ is increasing on the interval $[\sqrt{x_{+,*}}, +\infty[$. For this, we use the following representation of holomorphic matrix-valued functions whose imaginary part is positive definite on $\mathbb{C}^+$ (see e.g. [16]) :

$$H_*(z) = A + Bz + \int \frac{1 + \lambda z}{\lambda - z} \frac{d\sigma(\lambda)}{1 + \lambda^2} \tag{4.55}$$

where A is Hermitian, $B \geq 0$ and σ is a positive matrix-valued measure for which

$$\text{Tr}\left(\frac{d\sigma(\lambda)}{1 + \lambda^2}\right) < +\infty$$

We notice that $B = \lim_{y \to +\infty} \frac{H_*(iy)}{iy}$ coincides with

$$B = \lim_{y \to +\infty} \begin{pmatrix} -\frac{\mathbf{T}_{\beta_*}(iy)}{iy} & 0 \\ 0 & -\frac{\mathbf{T}_{\beta_*}(iy)}{iy} \end{pmatrix} = I_{2r}$$

and that for any interval $[y_1, y_2]$, it holds that

$$\sigma([y_1, y_2]) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{y_1}^{y_2} \text{Im}(H_*(y + i\epsilon)) dy$$

As $\text{Im}(H_*(y)) = 0$ if $|y| > \sqrt{x_{+,*}}$, the support of σ is included into $[-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$. Therefore, we get immediately from (4.55) that $y \to H_*(y)$ is strictly increasing on $]\sqrt{x_{+,*}}, +\infty]$, i.e. $H_*(y_2) > H_*(y_1)$ if $y_2 > y_1$. We also notice that the last item of Proposition 4.1 implies that $\lim_{y \to \sqrt{x_{+,*}}} H_*(y) = H_*(x_{+,*})$ exists and is finite. Moreover, it holds that $H_*(\sqrt{x_{+,*}}) < H_*(y)$ for $y > \sqrt{x_{+,*}}$.

Corollary 4.1. *The eigenvalues (arranged in the decreasing order) $(\lambda_{k,*}(y))_{k=1,\ldots,2r}$ of matrix $H_*(y)$ are strictly increasing functions y on $[\sqrt{x_{+,*}}, +\infty[$, i.e., for each $k = 1, \ldots, 2r$, it holds that*

$$\lambda_{k,*}(y_1) < \lambda_{k,*}(y_2) \text{ if } \sqrt{x_{+,*}} \leq y_1 < y_2 \tag{4.56}$$

Moreover, the number s of solutions of (4.53) (taking into account their multiplicities) for which $y > \sqrt{x_{+,*}}$ belongs to $\{0, 1, \ldots, 2r\}$, and coincides with the number of strictly negative eigenvalues of matrix $H_*(\sqrt{x_{+,*}})$.

Proof. We have shown that if $\sqrt{x_{+,*}} \leq y_1 < y_2$, then $H_*(y_1) < H_*(y_2)$. The Weyl's inequalities (see e.g. [24], Paragraph 4.3) thus imply that (4.56) holds. Moreover, as matrix B in (4.55) is equal to I_r, it is clear that for each $k = 1, \ldots, 2r$, $\lambda_{k,*}(y)$ converges towards $+\infty$ when $y \to +\infty$. For $k = 1, \ldots, 2r$, the equation $\lambda_{k,*}(y) = 0$ has thus 1 solution $y > \sqrt{x_{+,*}}$ if $\lambda_k(x_{+,*}) < 0$ and 0 solution if $\lambda_k(x_{+,*}) \geq 0$. (4.53) holds if and only one of the eigenvalues of $H_*(y)$ is equal to 0. Therefore, if we denote by $\tilde{s}$ the number of positive eigenvalues of $H_*(\sqrt{x_{+,*}})$, for $j = 1, \ldots, \tilde{s}$, it must hold that $\lambda_{j,*}(y) > 0$ for $y > \sqrt{x_{+,*}}$. Moroever, $\lambda_{\tilde{s}+1,*}(\sqrt{x_{+,*}}) < 0$ implies that the equation $\lambda_{\tilde{s}+1,*}(y) = 0$ has a unique solution $y_{1,*} > \sqrt{x_{+,*}}$. Similarly, the equation $\lambda_{\tilde{s}+2,*}(y) = 0$ has a unique solution denoted $y_{2,*}$. Moreover, as $\lambda_{\tilde{s}+2,*}(y) \leq \lambda_{\tilde{s}+1}(y)$ for each y, we deduce that $\lambda_{\tilde{s}+2,*}(y_{1,*}) \leq \lambda_{\tilde{s}+1}(y_{1,*}) = 0$. If $\lambda_{\tilde{s}+2,*}(y_{1,*}) < 0$, $y_{2,*}$ must be strictly greater than $y_{1,*}$. As a root of (4.53), $y_{1,*}$ has thus multiplicity 1. If $\lambda_{\tilde{s}+2,*}(y_{1,*}) = 0$, the multiplicity of $y_{1,*}$ as a root of (4.53) is at least equal to 2. Iterating the process, we obtain that the number of solutions s (taking into account the multiplicities) of (4.53) is equal to $s = 2r - \tilde{s}$. Moreover, solutions $y_{1,*}, \ldots, y_{s,*}$ satisfy $y_{1,*} \leq y_{2,*} \leq \ldots \leq y_{s,*}$.

Corollary 4.1 implies that Eq. $\det(I + F_*(y)) = 0$ has s $(0 \leq s \leq 2r)$ solutions $(y_{k,*})_{k=1,\ldots,s}$ strictly greater than $\sqrt{x_{+,*}}$. We recall that, almost surely, the sequence of functions $(F_N(z))_{N \geq 1}$ converges uniformly on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards $F_*(z)$. Using the arguments used in [4], we obtain immediately the following result.

Corollary 4.2. *Almost surely, for N large enough, Eq.* $\det(I + F_N(y)) = 0$ *has s solutions $y_{1,N} \leq y_{2,N} \ldots \leq y_{s,N}$ such that $y_{k,N} > \sqrt{x_{+,*}}$, and for each $k = 1, \ldots, s$, it holds that* $\lim_{N\to+\infty} y_{k,N} = y_{k,*}$.

We have thus established the Theorem :

Theorem 4.1. *Almost surely, for each N large enough, the s largest eigenvalues $\hat{\lambda}_{1,N} \geq \ldots \geq \hat{\lambda}_{s,N}$ of matrix $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ escape from the interval $[0, x_{+,*}]$, and converge towards $\rho_{1,*} \geq \ldots \geq \rho_{s,*} > x_{+,*}$ defined by $\rho_{k,*} = y_{s+1-k,*}^2$ for $k = 1, \ldots, s$.*

s and the limit eigenvalues $(\rho_{k,*})_{k=1,\ldots,s}$ depend on the limit distributions ω_* and β_* that are rather immaterial. It is thus more appropriate to evaluate the asymptotic behaviour of the largest eigenvalues of $W_{f,N}W_{p,N}^*W_{p,N}W_{f,N}^*$ by using the finite N deterministic equivalent of $H_*(z)$. We thus define function $H_N(z)$ by

$$H_N(z) = \begin{pmatrix} \frac{c_N \mathbf{t}_N(z))}{1-(c_N \mathbf{t}_N(z))^2}\Delta_N^2 - (\mathbf{T}_{\beta_N}(z))^{-1} & \frac{\Gamma_N^*}{(1-(c_N \mathbf{t}_N(z))^2} \\ \frac{\Gamma_N}{(1-(c_N \mathbf{t}_N(z))^2} & \frac{c_N \mathbf{t}_N(z)}{1-(c_N \mathbf{t}_N(z))^2}\Delta_N^2 - (\mathbf{T}_{\beta_N}(z))^{-1} \end{pmatrix} \tag{4.57}$$

For each $\delta > 0$, $H_N(z)$ is holomorphic in $\mathbb{C} - [-\sqrt{x_{+,*} + \delta}, \sqrt{x_{+,*} + \delta}]$ and converges uniformly on each compact subset of $\mathbb{C} - [-\sqrt{x_{+,*}}, \sqrt{x_{+,*}}]$ towards function $H_*(z)$. Using again the approach of [4], we obtain that, for each N large enough, the equation $\det(H_N(y)) = 0$ has s solutions $y_{1,N} \leq \ldots \leq y_{s,N}$ that satisfy $y_{k,N} - y_{k,*} \to 0$ when $N \to +\infty$. Moreover, the convergence of $x_{+,N}$ and $w_{+,N}$ towards $x_{+,*} = \phi_*(w_{+,*})$ and $w_{+,*} = w_*(x_{+,*})$ imply that $\mathbf{t}_N(x_{+,N})$ converge towards $\mathbf{t}_*(x_{+,*})$. Therefore, (4.38) leads to the following Corollary.

Corollary 4.3. *$H_N(\sqrt{x_{+,N}})$ converges towards $H_*(\sqrt{x_{+,*}})$. Moroever, for N large enough, s also coincides with the number of strictly negative eigenvalues of matrix $H_N(\sqrt{x_{+,N}})$. Finally, if we define $\rho_{k,N}$ by $\rho_{k,N} = y_{s+1-k,N}^2$ for $k = 1, \ldots, s$, then it holds that $\hat{\lambda}_{k,N} - \rho_{k,N} \to 0$ almost surely.*

Writing $\mathbf{t}_N(z)$ as $\mathbf{t}_N(z) = z t_N(z^2)$, and using the expression (3.132) of $t_N(z)$ in terms of $w_N(z)$, we obtain after some algebra that matrix $H_N(\sqrt{x_{+,N}})$ is given by

$$H_N(\sqrt{x_{+,N}}) = \left(1 + c_N \frac{1}{M}\mathrm{Tr}(R_N(w_{+,N}I - R_N)^{-1})\right) \begin{pmatrix} G_N(\sqrt{x_{+,N}}) & \Gamma_N^* \\ \Gamma_N & G_N(\sqrt{x_{+,N}}) \end{pmatrix} \tag{4.58}$$

where $G_N(\sqrt{x_{+,N}})$ is defined by

$$G_N(\sqrt{x_{+,N}}) = \frac{c_N w_{+,N}}{\sqrt{x_{+,N}}} \frac{1}{M}\mathrm{Tr}(R_N(w_{+,N}I - R_N)^{-1}) \left[\left(\Theta_N^*(I_L \otimes (w_{+,N}I - R_N)^{-1})\Theta_N\right)^{-1} - \Delta_N^2 \right] \tag{4.59}$$

As $\left(1 + c_N \frac{1}{M}\mathrm{Tr}(R_N(w_{+,N}I - R_N)^{-1})\right) > 0$, s coincides with the number of strictly negative eigenvalues of the second term of the right handside of (4.58).

In order to get some insights on the number of eigenvalues s that escape from $\mathcal{S}_N$ for each N large enough, we first study the behaviour of s when $c_N \to 0$. Intuitively, we should recover the results corresponding to the traditional regime, i.e. that $s = P$. For this, we remark that $w_{+,N}$, that depends on c_N, satisfies $\phi_N'(w_{+,N}) = 0$. Therefore, the proof of Proposition 3.10 implies that $\frac{1}{M}\mathrm{Tr}(R_N(w_{+,N}I - R_N)^{-1}) < 1$. As $R_N > aI$, we obtain that $a\frac{1}{M}\mathrm{Tr}R_N(w_{+,N}I - R_N)^{-1} < 1$, or equivalently

$$\frac{1}{M}\sum_{k=1}^{M} \frac{1}{w_{+,N} - \lambda_{k,N}} < \frac{1}{a}$$

for each c_N. This implies that $\liminf_{c_N \to 0} w_{+,N} - \lambda_{1,N} > 0$, and that matrix $\left(\Theta_N^*(I_L \otimes (w_{+,N}I - R_N)^{-1})\Theta_N\right)^{-1}$ remains bounded when $c_N \to 0$. As $x_{+,N} = \phi_N(w_{+,N})$, it is easy to check that $x_{+,N} = \mathcal{O}(c_N)$. Therefore, $\frac{c_N w_{+,N}}{\sqrt{x_{+,N}}} = \mathcal{O}(\sqrt{c_N})$, and $G_N(\sqrt{x_{+,N}}) \to 0$ when $c_N \to 0$. Therefore, when $c_N \to 0$,

$$H_N(\sqrt{x_{+,N}}) \to \begin{pmatrix} 0 & \Gamma_N^* \\ \Gamma_N & 0 \end{pmatrix}$$

As mentioned previously, matrix Γ_N has rank $P \leq r$. Therefore, the eigenvalues of matrix $\begin{pmatrix} 0 & \Gamma_N^* \\ \Gamma_N & 0 \end{pmatrix}$ are 0 with mutiplicity $2(r-P)$, $(\chi_k)_{k=1,\ldots,P}$ and $-(\chi_k)_{k=1,\ldots,P}$ where we recall that $(\chi_k)_{k=1,\ldots,P}$ represent the P non zero singular values of matrix Γ_N. Therefore, when $c_N \to 0$, s converges towards P. This is in accordance with the traditional asymptotic regime where $N \to +\infty$ and M is fixed. Indeed, in this context, matrix $\Sigma_{f,N}\Sigma_{p,N}^*\Sigma_{p,N}\Sigma_{f,N}^*$ converges towards the rank P matrix $R_{f|p}^L \left(R_{f|p}^L\right)^*$, i.e. for N large enough, matrix $\Sigma_{f,N}\Sigma_{p,N}^*\Sigma_{p,N}\Sigma_{f,N}^*$ has P eigenvalues that are significantly larger the $M-P$ smallest ones.

When c_N does not converge towards 0, the presence of matrix $G_N(\sqrt{x_{+,N}})$ in the expression (4.58) in general deeply modifies the value of s. In particular, the value of s depends on the singular values $(\chi_{k,N})_{k=1,\ldots,P}$ of matrix Γ_N, but also on the diagonal entries $(\delta_{k,N}^2)_{k=1,\ldots,r}$ of matrix Δ_N^2, or equivalently, on the non zero eigenvalues of $R_{u,N}^L = \mathbb{E}(u_n^L u_n^{*L})$. In particular, in contrast with the context of the usual spiked empirical covariance matrix models, s may be larger than the number P of non zero eigenvalues of the true matrix $R_{f|p}R_{f|p}^*$. This implies that if c_N is not small enough, then estimating the rank P of matrix $R_{f|p}R_{f|p}^*$ by the number s of eigenvalues of $\Sigma_{f,N}\Sigma_{p,N}^*\Sigma_{p,N}\Sigma_{f,N}^*$ that escape from $[0, x_{+,N}]$ does not lead to a consistant estimation scheme, even if the signal u is powerfull enough.

More precisely, assume that matrix $G_N(\sqrt{x_{+,N}})$ is invertible. Then, matrix $H_N(\sqrt{x_{+,N}})$ and the block diagonal matrix

$$\begin{pmatrix} G_N(\sqrt{x_{+,N}}) & 0 \\ 0 & G_N(\sqrt{x_{+,N}}) - \Gamma_N \left(G_N(\sqrt{x_{+,N}})\right)^{-1} \Gamma_N^* \end{pmatrix}$$

have the same number of strictly negative eigenvalues. If we denote by s_1 and s_2 the number of stricty negative eigenvalues of $G_N(\sqrt{x_{+,N}})$ and $G_N(\sqrt{x_{+,N}}) - \Gamma_N \left(G_N(\sqrt{x_{+,N}})\right)^{-1} \Gamma_N^*$ respectively, it holds that $s = s_1 + s_2$. In order to evaluate s_2, we denote by $\Upsilon_N^\perp$ a $r \times (r-P)$ matrix for which $(\Upsilon_N, \Upsilon_N^\perp)$ is unitary (we recall that Υ_N is defined by (4.13)). It is clear that s_2 coincides with the number of strictly negative eigenvalues of the block matrix

$$\begin{pmatrix} \Upsilon_N^* G_N \Upsilon_N - \Xi_N \tilde{\Upsilon}_N^* G_N^{-1} \tilde{\Upsilon}_N \Xi_N & \Upsilon_N^* G_N \Upsilon_N^\perp \\ \Upsilon_N^{\perp *} G_N \Upsilon_N & \Upsilon_N^{\perp *} G_N \Upsilon_N^\perp \end{pmatrix}$$

We have denoted $G_N(\sqrt{x_{+,N}})$ by G_N in order to simplify the notations. s_2 also coincides with the number of strictly negative eigenvalues of the block matrix

$$\begin{pmatrix} \Upsilon_N^* G_N \Upsilon_N - \Upsilon_N^* G_N \Upsilon_N^\perp \left(\Upsilon_N^{\perp *} G_N \Upsilon_N^\perp\right)^{-1} \Upsilon_N^{\perp *} G_N \Upsilon_N - \Xi_N \tilde{\Upsilon}_N^* G_N^{-1} \tilde{\Upsilon}_N \Xi_N & 0 \\ 0 & \Upsilon_N^{\perp *} G_N \Upsilon_N^\perp \end{pmatrix}$$

As it holds that

$$\Upsilon_N^* G_N \Upsilon_N - \Upsilon_N^* G_N \Upsilon_N^\perp \left(\Upsilon_N^{\perp *} G_N \Upsilon_N^\perp\right)^{-1} \Upsilon_N^{\perp *} G_N \Upsilon_N = \left(\Upsilon_N^* G_N^{-1} \Upsilon_N\right)^{-1}$$

s_2 is equal to $s_2 = s_{2,1} + s_{2,2}$ where $s_{1,2}$ (resp. $s_{2,2}$) represents the number of strictly negative eigenvalues of $\left(\Upsilon_N^* G_N^{-1} \Upsilon_N\right)^{-1} - \Xi_N \tilde{\Upsilon}_N^* G_N^{-1} \tilde{\Upsilon}_N \Xi_N$ (resp. of $\Upsilon_N^{\perp *} G_N \Upsilon_N^\perp$).

As it is difficult to evaluate precisely s in general cases, we focus on particular contexts. We first consider the case where matrix G_N is negative definite, i.e. $s_1 = r$. This condition holds if $\Delta_N^2 > \left(\Theta_N^* (I_L \otimes (w_{+,N} I - R_N)^{-1}) \Theta_N^*\right)^{-1}$ a condition which implies that all the entries of Δ_N^2 are large enough. In particular, it is easily seen that $G_N < 0$ as soon as $\delta_{r,N}^2 > w_{+,N} - \lambda_{M,N}$. As $\Upsilon_N^{\perp *} G_N \Upsilon_N^\perp < 0$, $s_{2,2}$ coincides with $r - P$. Therefore, if all the $(\delta_{k,N}^2)_{k=1,\ldots,r}$, i.e. if all the non zero eigenvalues of the covariance matrix $R_{u,N}^L$ are large enough, then $s = 2r - P + s_{2,1}$ and $s \geq 2r - P$. In order to discuss on the possible values of $s_{2,1}$, we denote by K_N the positive definite matrix $-G_N$. Then, $s_{2,1}$ coincides with the number of strictly positive eigenvalues of $\left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{-1} - \Xi_N \tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N \Xi_N$ or equivalently, of matrix

$$I_P - \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} \Xi_N \tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N \Xi_N \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2}$$

Therefore, s is equal to $2r$ if and only if

$$\left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} \Xi_N \tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N \Xi_N \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} < I_P \tag{4.60}$$

This condition holds if and only if matrix Ξ_N can be written as

$$\Xi_N = \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{-1/2} E_N \left(\tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N\right)^{-1/2} \tag{4.61}$$

where E_N verifies $\|E_N\| < 1$. This implies that for each $k = 1, \ldots, P$,

$$\chi_{k,N} = e_k^T \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} E_N \left(\tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N\right)^{-1/2} e_k$$

where $(e_k)_{k=1,\ldots,P}$ represents the canonical basis of $\mathbb{C}^P$. Therefore, for each k, it holds that

$$\chi_{k,N} < \|e_k^T \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} \| \| \left(\tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N\right)^{-1/2} e_k\|$$

or equivalently,

$$\chi_{k,N} < \left(e_k^T \Upsilon_N^* K_N^{-1} \Upsilon_N e_k\right)^{1/2} \left(e_k^T \tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N e_k\right)^{1/2}$$

Therefore, if the $(\delta_{k,N}^2)_{r=1,\ldots,r}$ are large enough, $s = 2r$ implies that the P non zero singular values $(\chi_{k,N})_{k=1,\ldots,P}$ of Γ_N have to be small enough. Conversely, s is reduced to $2r - P$ if and only Ξ_N can be written as

$$\Xi_N = \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{-1/2} F_N \left(\tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N\right)^{-1/2} \tag{4.62}$$

where F_N verifies $F_N F_N^* > I_P$. Therefore, for each $k = 1, \ldots, P$, it holds that

$$\chi_{k,N} = e_k^T \left(\Upsilon_N^* K_N^{-1} \Upsilon_N\right)^{1/2} F_N \left(\tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N\right)^{-1/2} e_k$$

and that

$$\chi_{k,N} > \left(e_k^T \Upsilon_N^* K_N^{-1} \Upsilon_N e_k\right)^{1/2} \left(e_k^T \tilde{\Upsilon}_N^* K_N^{-1} \tilde{\Upsilon}_N e_k\right)^{1/2}$$

Hence, (4.62) implies that all the singular values $(\chi_{k,N})_{k=1,\ldots,P}$ have to be large enough. Therefore, if the $(\delta_{k,N}^2)_{k=1,\ldots,r}$ are large enough, then s is equal to $2r - P$ if the singular values $(\chi_{k,N})_{k=1,\ldots,P}$ are large enough. In sum, when the non zero eigenvalues of the covariance matrix $R_{u,N}^L$ are large enough, $s \geq 2r - P$, and s is all the larger than the singular values $(\chi_{k,N})_{k=1,\ldots,P}$ of Γ_N are small, a rather non intuitive behaviour.

A different behaviour holds when matrix $G(\sqrt{x_{+,N}}) > 0$, i.e. $s_1 = 0$, a condition which is verified if all the non zero eigenvalues $(\delta_{k,N}^2)_{k=1,\ldots,P}$ are small enough. A sufficient condition is $\delta_{1,N}^2 < w_{+,N} - \lambda_{1,N}$. In this case, $s_1 = s_{2,2} = 0$ $s = s_{2,1} \leq P$. Moroever, $s_{2,1}$ coincides with the number of strictly negative eigenvalues of $\left(\Upsilon_N^* G_N^{-1} \Upsilon_N\right)^{-1} - \Xi_N \tilde{\Upsilon}_N^* G_N^{-1} \tilde{\Upsilon}_N \Xi_N$, or equivalently of matrix

$$I_P - \left(\Upsilon_N^* G_N^{-1} \Upsilon_N\right)^{1/2} \Xi_N \tilde{\Upsilon}_N^* G_N^{-1} \tilde{\Upsilon}_N \Xi_N \left(\Upsilon_N^* G_N^{-1} \Upsilon_N\right)^{1/2}$$

Using the same approach as when $G_N < 0$, we obtain that $s_{2,1} = 0$, i.e. $s = 0$ implies that the $(\chi_{k,N})_{k=1,\ldots,P}$ are small enough, while $s = s_{2,1} = P$ if the $(\chi_{k,N})_{k=1,\ldots,P}$ are large enough. In sum, if the $(\delta_{k,N}^2)_{k=1,\ldots,P}$ are small enough, $s = 0$ if all the $(\chi_{k,N})_{k=1,\ldots,P}$ are small enough, while $s = P$ if the $(\chi_{k,N})_{k=1,\ldots,P}$ are large enough. We however notice that the $(\chi_{k,N})_{k=1,\ldots,P}$ and the $(\delta_{k,N}^2)_{k=1,\ldots,P}$ are not independent parameters. In particular, it holds that $\|\Gamma_N\| \leq \delta_{1,N}^2$, and therefore that $\chi_{k,N} \leq \delta_{1,N}^2$ for each $k = 1, \ldots, P$. Therefore, the conditions that $(\delta_{k,N}^2)_{k=1,\ldots,P}$ are small enough and $(\chi_{k,N})_{k=1,\ldots,P}$ are large enough may not be both verified.

In order to get more insights on the above discussion, we consider the simple case where $P = 1$ and $R_N = \sigma^2 I_M$. $w_{+,N}$ and $x_{+,N}$ are given by (3.143) and (3.142). Moreover, for each $w \geq w_{+,N}$, $x = \phi_N(w) \geq x_{+,N}$ and $\frac{(c_N w)^2}{x} \frac{1}{M} \mathrm{Tr}(R_N(wI - R_N)^{-1})$ is equal to

$$\frac{(c_N w)^2}{x} \frac{1}{M} \mathrm{Tr}(R_N(wI - R_N)^{-1} = \frac{\sigma^2 c_N}{w - \sigma^2(1 - c_N)}$$

Matrix G_N is thus given by

$$G_N = \left(\frac{\sigma^2 c_N}{w_{+,N} - \sigma^2(1-c_N)} \right)^{1/2} \left((w_{+,N} - \sigma^2) I_r - \Delta_N^2 \right)$$

As $P = 1$, Γ_N is a rank 1 matrix. Matrices Υ_N and $\tilde{\Upsilon}_N$ are reduced to r–dimensional vectors, and diagonal matrix Ξ_N is reduced to a scalar χ_N.
We first consider the case where $G_N < 0$. This condition holds if and only if $\delta_{r,N}^2 > w_{+,N} - \sigma^2$. Our results show that s is equal to $2r$ if

$$1 - \chi_N^2 \frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \sum_{k=1}^r \frac{|\Upsilon|_k^2}{\delta_{k,N}^2 - (w_{+,N} - \sigma^2)} \sum_{k=1}^r \frac{|\tilde{\Upsilon}|_k^2}{\delta_{k,N}^2 - (w_{+,N} - \sigma^2)} > 0$$

while $s = 2r - 1$ if

$$1 - \chi_N^2 \frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \sum_{k=1}^r \frac{|\Upsilon|_k^2}{\delta_{k,N}^2 - (w_{+,N} - \sigma^2)} \sum_{k=1}^r \frac{|\tilde{\Upsilon}|_k^2}{\delta_{k,N}^2 - (w_{+,N} - \sigma^2)} < 0$$

When $G_N > 0$, i.e. if $\delta_{1,N}^2 < w_{+,N} - \sigma^2$, $s = 0$ or $s = 1$. More precisely, $s = 0$ if and only if

$$1 - \chi_N^2 \frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \sum_{k=1}^r \frac{|\Upsilon|_k^2}{(w_{+,N} - \sigma^2) - \delta_{k,N}^2} \sum_{k=1}^r \frac{|\tilde{\Upsilon}|_k^2}{(w_{+,N} - \sigma^2) - \delta_{k,N}^2} > 0$$

while $s = 1$ if

$$1 - \chi_N^2 \frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \sum_{k=1}^r \frac{|\Upsilon|_k^2}{(w_{+,N} - \sigma^2) - \delta_{k,N}^2} \sum_{k=1}^r \frac{|\tilde{\Upsilon}|_k^2}{(w_{+,N} - \sigma^2) - \delta_{k,N}^2} < 0$$

In order to confirm this behaviour, we evaluate more directly the value of s by studying directly the solutions of the equation $\det(H_N(\sqrt{x})) = 0$, or equivalently the solutions of

$$\det \begin{pmatrix} G_N(\sqrt{x}) & \Gamma_N^* \\ \Gamma_N & G_N(\sqrt{x}) \end{pmatrix} = 0 \tag{4.63}$$

that are larger than $x_{+,N}$. In order to evaluate the solutions of (4.63), we establish the following Lemma.

Lemma 4.2. *$x = \phi_N(\sigma^2 + \delta_{l,N}^2)$ is solution of (4.63) if and only $\Upsilon_l = 0$ or $\tilde{\Upsilon}_l = 0$. Moreover, if for each $k = 1, \ldots, r$, Υ_k and $\tilde{\Upsilon}_k$ are non zero, then (4.63) holds if and only if*

$$1 - \chi_N^2 \left(\Upsilon^* G_N(\sqrt{x})^{-1} \Upsilon \right) \left(\tilde{\Upsilon}^* G_N(\sqrt{x})^{-1} \tilde{\Upsilon} \right) = 0 \tag{4.64}$$

Proof. If w is not equal to $\sigma^2(1-c_N), \sigma^2 + \delta_{1,N}^2, \ldots, \sigma^2 + \delta_{r,N}^2$, the left handside of (4.63) can be written as

$$\det \begin{pmatrix} G_N(\sqrt{x}) & \Gamma_N^* \\ \Gamma_N & G_N(\sqrt{x}) \end{pmatrix} = \det \left(G_N(\sqrt{x}) \right) \det \left(G_N(\sqrt{x}) - \Gamma_N G_N(\sqrt{x})^{-1} \Gamma_N^* \right)$$

Moreover, it holds that

$$\det \left(G_N(\sqrt{x}) - \Gamma_N G_N(\sqrt{x})^{-1} \Gamma_N^* \right) = \det \left(\Upsilon^{\perp *} G_N(\sqrt{x}) \Upsilon^{\perp} \right) \left(\frac{1}{\Upsilon^* G_N(\sqrt{x})^{-1} \Upsilon} - \chi_N^2 \left(\tilde{\Upsilon}^* G_N(\sqrt{x})^{-1} \tilde{\Upsilon} \right) \right)$$

where we recall that $\Upsilon_N^{\perp}$ is a $r \times (r-1)$ orthogonal matrix such that $(\Upsilon_N, \Upsilon_N^{\perp})$ is a unitary matrix. Therefore, if w is not equal to $\sigma^2(1-c_N), \sigma^2 + \delta_{1,N}^2, \ldots, \sigma^2 + \delta_{r,N}^2$, it holds that

$$\det \begin{pmatrix} G_N(\sqrt{x}) & \Gamma_N^* \\ \Gamma_N & G_N(\sqrt{x}) \end{pmatrix} = \det(G_N(\sqrt{x})) \det(\Upsilon^{\perp *} G_N(\sqrt{x}) \Upsilon^{\perp}) \left(\frac{1}{\Upsilon^* G_N(\sqrt{x})^{-1} \Upsilon} - \chi_N^2 \left(\tilde{\Upsilon}^* G_N(\sqrt{x})^{-1} \tilde{\Upsilon} \right) \right) \tag{4.65}$$

(4.65) still holds true when w coincides with one of the $(\sigma^2+\delta_{k,N}^2)_{k=1,\ldots,r}$ because for each $l=1,\ldots,r$, the right handside of (4.65) has a finite limit when $w \to \sigma^2+\delta_{l,N}^2$. More precisely,

$$\frac{1}{\Upsilon^* G_N(\sqrt{x})^{-1}\Upsilon} = \left(\frac{w-\sigma^2(1-c_N)}{\sigma^2 c_N}\sum_{k=1}^{r}\frac{|\Upsilon|_k^2}{\delta_{k,N}^2-(w-\sigma^2)}\right)^{-1} \to 0$$

and

$$\det(G_N(\sqrt{x}))\chi_N^2\left(\tilde{\Upsilon}^* G_N(\sqrt{x})^{-1}\tilde{\Upsilon}\right) \to \chi_N^2\left|\tilde{\Upsilon}_{l,N}\right|^2 \Pi_{k\neq l}(\delta_{l,N}^2-\delta_{k,N}^2)\left(\frac{\sigma^2 c_N}{\delta_{l,N}^2+\sigma^2 c_N}\right)^{r/2}$$

Moreover, if we denote by $\Upsilon_N^{\perp,l}$ the $(r-1)\times(r-1)$ matrix obtained by deleting the l-th row of $\Upsilon_N^{\perp}$,

$$\det(\Upsilon^{\perp *} G_N(\sqrt{x})\Upsilon^{\perp}) \to \det\left((\Upsilon_N^{\perp,l})^*\Upsilon_N^{\perp,l}\right)\Pi_{k\neq l}(\delta_{l,N}^2-\delta_{k,N}^2)\left(\frac{\sigma^2 c_N}{\delta_{l,N}^2+\sigma^2 c_N}\right)^{(r-1)/2}$$

This implies that the right handside of (4.65) converges towards

$$-\chi_N^2\left|\tilde{\Upsilon}_{l,N}\right|^2 \det\left((\Upsilon_N^{\perp,l})^*\Upsilon_N^{\perp,l}\right)\Pi_{k\neq l}(\delta_{l,N}^2-\delta_{k,N}^2)^2\left(\frac{\sigma^2 c_N}{\delta_{l,N}^2+\sigma^2 c_N}\right)^{r-1/2}$$

which, of course, coincides with $\det\begin{pmatrix} G_N(\sqrt{x}) & \Gamma_N^* \\ \Gamma_N & G_N(\sqrt{x})\end{pmatrix}$ for $x=\phi_N(\delta_{l,N}^2+\sigma^2)$. We denote by $\Upsilon_{l,N}^{\perp}$ the l-row of $\Upsilon_N^{\perp}$. Then, as $\Upsilon_N^{\perp}$ is orthogonal, it holds that $\Upsilon_{l,N}^{\perp *}\Upsilon_{l,N}^{\perp}+\left(\Upsilon_N^{\perp,l}\right)^*\Upsilon_N^{\perp,l}=I_{r-1}$. Therefore,

$$\det\left((\Upsilon_N^{\perp,l})^*\Upsilon_N^{\perp,l}\right)=\det(I_{r-1}-\Upsilon_{l,N}^{\perp *}\Upsilon_{l,N}^{\perp}) = 1-\|\Upsilon_{l,N}^{\perp}\|^2$$

As matrix $(\Upsilon_N, \Upsilon_N^{\perp})$ is unitary, it holds that $|\Upsilon_{l,N}|^2+\|\Upsilon_{l,N}^{\perp}\|^2=1$. Therefore, we obtain that if $x=\phi_N(\sigma^2+\delta_{l,N}^2)$, the left handside of (4.63) is equal to

$$-\chi_N^2\,|\tilde{\Upsilon}_{l,N}|^2\,|\Upsilon_{l,N}|^2\,\Pi_{k\neq l}(\delta_{l,N}^2-\delta_{k,N}^2)^2\left(\frac{\sigma^2 c_N}{\delta_{l,N}^2+\sigma^2 c_N}\right)^{r-1/2}$$

(4.63) thus holds for $x=\phi_N(\sigma^2+\delta_{l,N}^2)$ if and only if $|\tilde{\Upsilon}_{l,N}|^2\,|\Upsilon_{l,N}|^2=0$. If $|\tilde{\Upsilon}_{k,N}|^2\,|\Upsilon_{k,N}|^2>0$ for each $k=1,\ldots,r$, none of the $(\phi_N(\sigma^2+\delta_{k,N}^2))_{k=1,\ldots,r}$ is a solution of (4.63). Moroever, as $\det(G_N(\sqrt{x}))\det(\Upsilon^{\perp *}G_N(\sqrt{x})\Upsilon^{\perp})\neq 0$ if x does not belong to $\{(\phi_N(\sigma^2+\delta_{1,N}^2),\ldots,(\phi_N(\sigma^2+\delta_{r,N}^2)\}$, we obtain that $x>x_{+,N}$ is a solution of (4.63) if and only (4.64) holds. This completes the proof the Lemma.

In order to simplify the following discussion, we assume that for each $k=1,\ldots,r$, then $|\tilde{\Upsilon}_{k,N}|^2\,|\Upsilon_{k,N}|^2>0$. Lemma 4.2 implies that the eigenvalues of matrix $\Sigma_{f,N}\Sigma_{p,N}^*\Sigma_{p,N}\Sigma_{f,N}^*$ that escape from $[0,x_{+,*}]$, or equivalently from $[0,x_{+,N}]$, have the same asymptotic behaviour than the image by function ϕ_N of the solutions of the equation

$$\frac{w-\sigma^2(1-c_N)}{\sigma^2 c_N}\sum_{k=1}^{r}\frac{|\Upsilon|_k^2}{\delta_{k,N}^2-(w-\sigma^2)}\sum_{k=1}^{r}\frac{|\tilde{\Upsilon}|_k^2}{\delta_{k,N}^2-(w-\sigma^2)}=\frac{1}{\chi_N^2}$$

that are strictly larger than $w_{+,N}$. We consider the function $f_N(w)$ defined by

$$f_N(w)=\frac{w-\sigma^2(1-c_N)}{\sigma^2 c_N}\sum_{k=1}^{r}\frac{|\Upsilon|_k^2}{(w-\sigma^2)-\delta_{k,N}^2}\sum_{k=1}^{r}\frac{|\tilde{\Upsilon}|_k^2}{(w-\sigma^2)-\delta_{k,N}^2}$$

On each interval $]\sigma^2+\delta_{k,N}^2,\sigma^2+\delta_{k+1,N}^2[$ $(k=1,\ldots,r-1)$, functions $w\to\sum_{k=1}^{r}\frac{|\Upsilon|_k^2}{(w-\sigma^2)-\delta_{k,N}^2}$ and $w\to\sum_{k=1}^{r}\frac{|\tilde{\Upsilon}|_k^2}{(w-\sigma^2)-\delta_{k,N}^2}$ have each a unique zero. Therefore, f_N admits 2 zeros on $]\sigma^2+\delta_{k,N}^2,\sigma^2+\delta_{k+1,N}^2[$. Moreover,

f_N converges towards $+\infty$ when $w \to \sigma^2+\delta^2_{k,N}$ and when $w \to \sigma^2+\delta^2_{k+1,N}$. Therefore, for each $k = 1, \ldots, r-1$, the equation $f_N(w) = \frac{1}{\chi^2_N}$ has at least 2 solutions that belong to $]\sigma^2 + \delta^2_{k,N}, \sigma^2 + \delta^2_{k+1,N}[$. As $f_N(w) \to 0$ when $w \to +\infty$, the equation $f_N(w) = \frac{1}{\chi^2_N}$ has at least 1 solution that belongs to $]\sigma^2 + \delta^2_{1,N}, +\infty[$. $f_N(w) < 0$ if $w < \sigma^2(1 - c_N)$ and $f_N(w) \to +\infty$ when $w \to \sigma^2 + \delta^2_{1,N}$. The equation $f_N(w) = \frac{1}{\chi^2_N}$ has thus at least one solution that belongs to $]\sigma^2(1 - c_N), \sigma^2 + \delta^2_{1,N}[$. This discussion shows that $f_N(w) = \frac{1}{\chi^2_N}$ has at least $2r$ solutions that belong to $]\sigma^2(1 - c_N), +\infty[$. It is moreover easily seen that this equation is a polynomial equation of order $2r$. Therefore, the $2r$ real roots of $f_N(w) = \frac{1}{\chi^2_N}$ located in $]\sigma^2(1 - c_N), +\infty[$ coincide with the set of all roots of the equation. This also implies that f_N is strictly increasing on $]\sigma^2(1 - c_N), \delta^2_{r,N} + \sigma^2[$ and strictly decreasing on $]\sigma^2 + \delta^2_{1,N}, +\infty[$. Moreover, on $]\sigma^2 + \delta^2_{k,N}, \sigma^2 + \delta^2_{k+1,N}[$, f_N is first decreasing, and then increasing. This discussion leads to the conclusion that s coincides with the number of roots that are strictly larger than $w_{+,N}$. To connect with the above results, we remark that :

— If $\delta^2_{r,N} > w_{+,N} - \sigma^2$, i.e. if $G_N(\sqrt{x_{+,N}}) < 0$, $2r-1$ roots of $f_N(w) = \frac{1}{\chi^2_N}$ belong to $]\delta^2_{r,N} + \sigma^2, +\infty[$, and are therefore larger than $w_{+,N}$. It thus holds that $s \geq (2r - 1)$. $s = 2r$ if and only if $\frac{1}{\chi^2_N} > f_N(w_{+,N})$, i.e. $\chi^2_N < \frac{1}{f_N(w_{+,N})}$, and $s = 2r - 1$ if and only if $\chi^2_N > \frac{1}{f_N(w_{+,N})}$. This is of course in accordance with the evaluation of s based on the number of strictly negative eigenvalues of matrix $H_N(\sqrt{x_{+,N}})$.

— If $\delta^2_{1,N} < w_{+,N} - \sigma^2$, i.e. if $G_N(\sqrt{x_{+,N}}) > 0$, the $(2r - 1)$ roots of $f_N(w) = \frac{1}{\chi^2_N}$ that are located in $]\sigma^2(1 - c_N), \delta^2_{1,N} + \sigma^2[$ are smallest that $w_{+,N}$. Therefore, $s = 0$ or $s = 1$. As f_N is decreasing on $]\delta^2_{1,N} + \sigma^2, +\infty[$, $s = 0$ if and only $\frac{1}{\chi^2_N} < f_N(w_{+,N})$, i.e. if $\chi^2_N < \frac{1}{f_N(w_{+,N})}$, and $s = 1$ if $\chi^2_N > \frac{1}{f_N(w_{+,N})}$ as expected.

In the simple case $P = 1$, it is even possible to precise the value of s when $\delta^2_{l+1,N} + \sigma^2 < w_{+,N} < \delta^2_{l,N} + \sigma^2$ for some $l = 1, \ldots, r - 1$. The equation $f_N(w) = \frac{1}{\chi^2_N}$ has $2(l - 1) + 1 = 2l - 1$ solutions belonging to $]\delta^2_{l,N} + \sigma^2, +\infty[$. Therefore, $s \geq 2l - 1$. If we denote by $w_{1,l} \leq w_{2,l}$ the 2 zeros of function f_N located in $]\delta^2_{l+1,N} + \sigma^2, \delta_{l,N} + \sigma^2[$, it is moreover easy to check that :

— if $w_{+,N} < w_{1,l}$, then $s = 2l + 1$ if and only if $f(w_{+,N}) > \frac{1}{\chi^2_N}$, and $s = 2l$ otherwise

— if $w_{+,N} > w_{2,l}$, $s = 2l$ if and only if $f_N(w_{+,N}) < \frac{1}{\chi^2_N}$ and $s = 2l - 1$ otherwise

It is also possible to understand how the eigenvalues of $\Sigma_{f,N}\Sigma^*_{p,N}\Sigma_{p,N}\Sigma^*_{f,N}$ that escape from $[0, x_{+,N}]$ behave when c_N tends to be very small. As mentioned previously, in the standard asymptotic regime where $N \to +\infty$ while M remains fixed, the smallest $ML - 1$ eigenvalues of $\Sigma_{f,N}\Sigma^*_{p,N}\Sigma_{p,N}\Sigma^*_{f,N}$ converge towards 0, and its largest eigenvalue has the same behaviour than χ^2_N. When c_N takes small values, this behaviour should be observed. Assume for example that $G_N < 0$, and that $s = 2r$, i.e. that $f_N(w_{+,N}) < \frac{1}{\chi^2_N}$. Then, the $2r - 1$ smallest solutions of $f_N(w) = \frac{1}{\chi^2_N}$, denoted $w_{2,N} > w_{3,N} \ldots > w_{2r,N}$, belong to $]w_{+,N}, \delta^2_{1,N} + \sigma^2[$ while the largest solution, $w_{1,N}$ is located into $]\delta^2_{1,N} + \sigma^2, +\infty[$. The $2r - 1$ smallest eigenvalues of $\Sigma_{f,N}\Sigma^*_{p,N}\Sigma_{p,N}\Sigma^*_{f,N}$ that escape from $[0, x_{+,N}]$ behave as the $(x_{l,N} = \phi_N(w_{l,N}))_{l=2,\ldots,2r}\}$ and the largest one as $x_{1,N} = \phi_N(w_{1,N})$. It is clear that $x_{l,N} \leq \phi_N(\sigma^2 + \delta^2_{1,N})$ for $l = 2, \ldots, 2r$. We recall that $\phi_N(w)$ is given by

$$\phi_N(w) = c_N w^2 \frac{\sigma^2}{w - \sigma^2}\left(1 + \frac{\sigma^2 c_N}{w - \sigma^2}\right)$$

As $(\delta^2_{1,N})_{N\geq 1}$ is assumed to be bounded, it holds that if $l = 2, \ldots, 2r$, $x_{l,N} = \mathcal{O}(c_N)$ when $c_N \to 0$. Therefore, the $2r - 1$ smallest eigenvalues of $\Sigma_{f,N}\Sigma^*_{p,N}\Sigma_{p,N}\Sigma^*_{f,N}$ that escape from $[0, x_{+,N}]$ converge towards 0 at rate c_N when $c_N \to 0$. In order to evaluate the behaviour of $x_{1,N}$ when $c_N \to 0$, we recall that $w_{1,N} > \delta^2_{1,N} + \sigma^2$ satisfies

$$\frac{w_{1,N} - \sigma^2(1 - c_N)}{\sigma^2 c_N} \sum_{k=1}^{r} \frac{|\Upsilon|^2_k}{(w_{1,N} - \sigma^2) - \delta^2_{k,N}} \sum_{k=1}^{r} \frac{|\tilde{\Upsilon}|^2_k}{(w_{1,N} - \sigma^2) - \delta^2_{k,N}} = \frac{1}{\chi^2_N}$$

When $c_N \to 0$, $w_{1,N}$ has clearly to converge towards $+\infty$. Moreover, it is easily seen that $w_{1,N} \simeq \frac{\chi^2_N}{\sigma^2 c_N}$ when $c_N \to 0$. Therefore, $x_{1,N} = \phi_N(w_{1,N})$ is itself equivalent to χ^2_N as expected. Therefore, when c_N takes small values, the $2r - 1$ smallest eigenvalues that escape from $[0, x_{+,N}]$ are $\mathcal{O}(c_N)$ terms, while the largest eigenvalue tends to be close from χ_N^2. The same conclusions hold if G_N is not negative definite.

We finally conclude this discussion by an even simpler case. We assume that $P = K = 1$, and that the scalar state-space sequence $(x_n)_{n\in\mathbb{Z}}$ is given by $x_{n+1} = ax_n + b\nu_n$ where $a \in]0,1[$ and $b \in \mathbb{C}$. Moroever, u_n is given by

$$u_n = \theta_N x_{n+1} = a\theta_N x_n + b\theta_N \nu_n \tag{4.66}$$

where θ_N is a unit norm M–dimensional vector. Therefore, matrices C_N and D_N coincide with vectors $a\theta_N$ and $b\theta_N$ respectively. We also consider the case where $L = 1$. Model (4.66) fits into the framework of [29]. In this context, matrix $U_{f,N}$ and $U_{p,N}$ are given by

$$U_{f,N} \; = \theta_N \, (x_3, x_4, \ldots, x_{N+2}), U_{p,N} = \theta_N \, (x_2, x_3, \ldots, x_{N+1})$$

The covariance matrix $E(u_n u_n^*)$ is of course equal to $E(u_n u_n^*) = \delta^2 \, \theta_N \theta_N^*$ where $\delta^2 = \mathbb{E}(|x_n|^2) = \frac{|b|^2}{1-a^2}$, so that $r = P = 1$. We also mention that in the present case, δ^2 does not depend on N. Moreover, the empirical autocovariance matrix $U_{f,N}U_{p,N}^*/N$ coincides with

$$U_{f,N}U_{p,N}^*/N = \hat{r}_{x,N}(1) \, \theta_N \theta_N^*$$

where $\hat{r}_{x,N}(1) = \frac{1}{N}\sum_{k=2}^{N+1} x_{k+1}x_k^*$ is the traditional empirical estimate of $E(x_{n+1}x_n^*) = a\delta^2$. Therefore, the usual $r \times r$ matrix Γ_N is reduced to the scalar $\hat{r}_{x,N}(1)$, and the associated singular value of Γ_N is $\chi_N = |\hat{r}_{x,N}(1)|$. We notice that $\hat{r}_{x,N}(1)$, and therefore χ_N, converge towards $a\delta^2$ when $N \to +\infty$. As $r = P = 1$, s may take the values $0, 1, 2$. In the following, we justify that it is possible to find a and b for which the above 3 possible values of s are possible.

We first find a and b for which $s = 2$. $s = 2$ if and only if $\delta^2 > w_{+,N} - \sigma^2$ and $1 - \chi_N^2 f_N(w_{+,N}) > 0$, or equivalently

$$1 - \chi_N^2 \left(\frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \right) \left(\frac{1}{\delta^2 - (w_{+,N} - \sigma^2)} \right)^2 > 0$$

As χ_N can be arbitrarily close from $a\delta^2$ for N large enough, this last condition can be replaced by

$$1 - a^2\delta^4 \left(\frac{w_{+,N} - \sigma^2(1-c_N)}{\sigma^2 c_N} \right) \left(\frac{1}{\delta^2 - (w_{+,N} - \sigma^2)} \right)^2 > 0$$

or equivalently by

$$a^2 < \frac{\sigma^2 c_N}{\sigma^2 c_N + (w_{+,N} - \sigma^2)} \left(1 - \frac{(w_{+,N} - \sigma^2}{\delta^2} \right)^2 \tag{4.67}$$

In order to find $a \in]0,1[$ and b for which these conditions hold, we fix $\delta^2 > w_{+,N} - \sigma^2$, then choose $a \in]0,1[$ such that (4.67) holds, and finally select b in such a way that $|b|^2 = \delta^2(1-a^2)$.

We now produce values of a and b for which $s = 1$ and $\delta^2 > w_{+,N} - \sigma^2$. For this, it is sufficient to find $a \in]0,1[$ and $\delta^2 > w_{+,N} - \sigma^2$ such that

$$1 > a^2 > \frac{\sigma^2 c_N}{\sigma^2 c_N + (w_{+,N} - \sigma^2)} \left(1 - \frac{w_{+,N} - \sigma^2}{\delta^2} \right)^2 \tag{4.68}$$

For each $\delta^2 > w_{+,N} - \sigma^2$, such an a exists because the term $\frac{\sigma^2 c_N}{\sigma^2 c_N + (w_{+,N} - \sigma^2)} \left(1 - \frac{w_{+,N} - \sigma^2}{\delta^2} \right)^2$ is strictly less than 1. Therefore, we consider any $\delta^2 > w_{+,N} - \sigma^2$, select $a \in]0,1[$ verifying (4.68), and finally choose b such that $|b|^2 = \delta^2(1-a^2)$.

We now show the existence of (a,b) for which $\delta^2 < w_{+,N} - \sigma^2$ and $s = 1$. For this, (4.68) has to be verified, i.e. it must exist $\delta^2 < w_{+,N} - \sigma^2$ such that

$$1 > \frac{\sigma^2 c_N}{\sigma^2 c_N + (w_{+,N} - \sigma^2)} \left(1 - \frac{w_{+,N} - \sigma^2}{\delta^2} \right)^2 \tag{4.69}$$

It is easy to check that δ^2 verifies (4.69) if and only if δ^2 is chosen in such a way that

$$w_{+,N} - \sigma^2 > \delta^2 > \frac{w_{+,N} - \sigma^2}{1 + \left(\frac{\sigma^2 c_N + w_{+,N} - \sigma^2}{\sigma^2 c_N}\right)^{1/2}}$$

We thus choose such a value for δ^2, choose a for which (4.69) holds, and finally select b in such a way that $|b|^2 = \delta^2(1-a^2)$.

Finally, $s = 0$ if and only if (4.67) and $\delta^2 < w_{+,N} - \sigma^2$ hold. We simply choose $\delta^2 < w_{+,N} - \sigma^2$, select $a \in]0,1[$ such that (4.67) holds, and choose b in such a way that $|b|^2 = \delta^2(1-a^2)$.

We finally illustrate the above discussion by numerical experiments showing that s can indeed by equal to $0, 1$ or 2. The particular values of a and b are not mentioned. Figures 4.1, 4.2, 4.3 represent histograms of the eigenvalues of realizations of the matrix $\Sigma_f \Sigma_p^* \Sigma_p \Sigma_f^*$ as well as the graph of the density g_N of measure ν_N. Figure 4.1 corresponds to a choice of a, b for which $s = 0$, while $s = 1$ and $s = 2$ in the context of Figures 4.2 and 4.3 respectively.

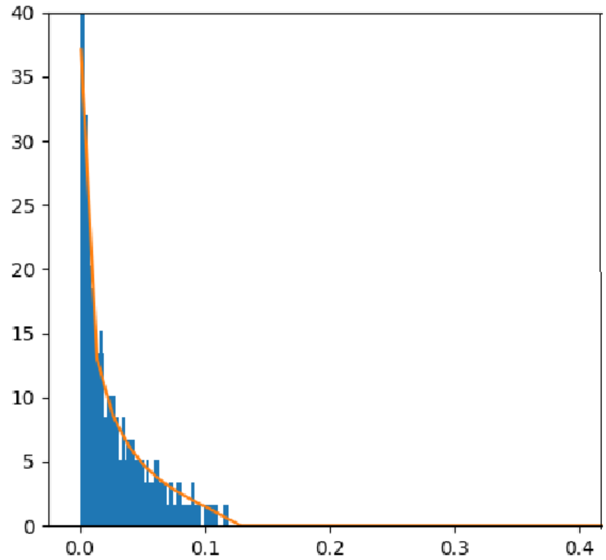

FIGURE 4.1 – Histogram of the eigenvalues and graph of g_N, $s = 0$

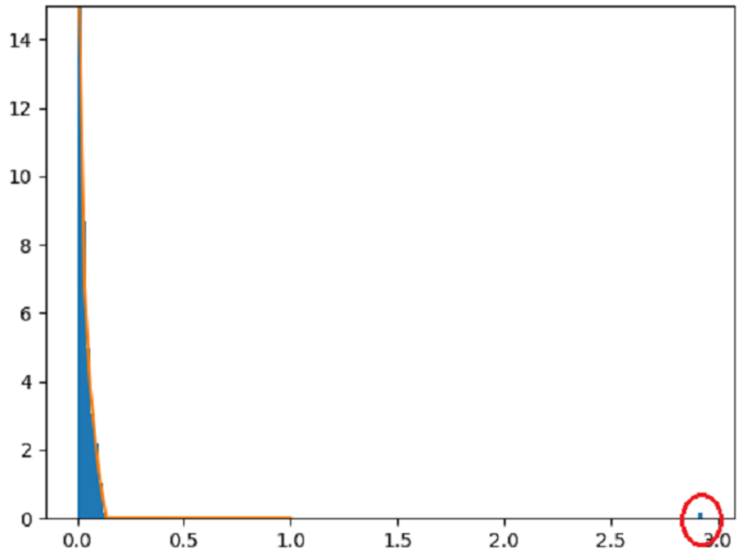

FIGURE 4.2 – Histogram of the eigenvalues and graph of g_N, $s = 1$

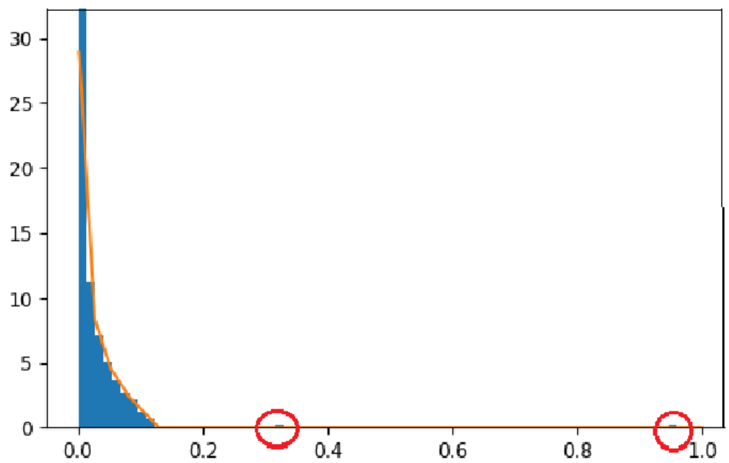

FIGURE 4.3 – Histogram of the eigenvalues and graph of g_N, $s = 2$

Chapitre 5

The canonical correlation coefficients between the past and the future

As we can see from the previous chapter in the high dimensional regime the number of eigenvalues of $\hat{R}_{f|p}\hat{R}^*_{f|p}$ that escape from the support is a bad estimator for dimension P of the minimal state space representation (1.2). So in this chapter we consider a different approach that is used in order to estimate P.
The canonical correlation coefficients are defined in time series analysis in order to evaluate the relationships between the past and the future of a given multivariate time series $(y_n)_{n\in\mathbb{Z}}$ (see e.g. [25]). In this context, we define the 2 subspaces, denoted $\mathcal{Y}_p$ (the past) and $\mathcal{Y}_f$ (the future), as the spaces generated by the components of y_n for $n \leq 0$ and the components of y_n for $n > 0$ respectively. We recall that if $(\omega p, k)_{k\geq 0}$ and $(\omega_{f,k})_{k\geq 0}$ represent orthonormal bases of $\mathcal{Y}_p$ and $\mathcal{Y}_f$, the canonical correlation coefficients between the past and the future of y are defined as the singular values of the infinite matrix with entries $\mathbb{E}\{\omega_{f,k}\omega^*_{p,l}\}$. In the case when y_n has a rational spectrum, the number of non zero canonical correlation coefficients between the past and the future of $(y_n)_{n\in\mathbb{Z}}$ is finite, and coincides with the minimal dimension P of the state-space representations of y. We refer the reader to [30] for an exhaustive presentation of the related results and their important implications on questions such as the identification of state space models or on reduction model technics. See also the concise monography [45]. In a number of practical procedures, $\mathcal{Y}_p$ and $\mathcal{Y}_f$ are replaced by the finite dimensional spaces $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ generated respectively by the components of $y_n, n = -(L-1), \dots, 0$ and $y_n, n = 1, \dots, L$ for a certain integer $L \geq P$, a condition that implies that the number of non zero coefficients between $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ is still equal to P. We refer again to [30] for more details on the effects of the truncation. As the second order statistics of y are very often unknown, the correlation coefficients between $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ have to be estimated from N available samples $y_1, \dots, y_N$. The correlation coefficients between $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$ are usually estimated by the canonical correlation coefficients between the row spaces of $Y_{p,L}$ and $Y_{f,L}$, which are define as above, i.e.

$$Y_{p,N} = \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-1} & y_N \\ y_2 & y_3 & \cdots & y_N & y_{N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_L & y_{L+1} & \cdots & y_{N+L-2} & y_{N+L-1} \end{pmatrix} \tag{5.1}$$

and

$$Y_{f,N} = \begin{pmatrix} y_{L+1} & y_{L+2} & \cdots & y_{N-1+L} & y_{N+L} \\ y_{L+2} & y_{L+3} & \cdots & y_{N+L} & y_{N+L+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{2L} & y_{2L+1} & \cdots & y_{N+2L-2} & y_{N+2L-1} \end{pmatrix}. \tag{5.2}$$

The above estimation procedure produces reasonably accurate results when the ratio $c_N = ML/N$ is small enough. However, if y is high-dimensional, i.e. if M is large, the condition $c_N << 1$ will not be verified as soon as the number of observations is not unlimited. It is therefore important to evaluate the behaviour of the above estimators when c_N is not negligible. In this chapter, we address this problem when y_n is generated as in Chapter 4 by studying the behaviour of the above estimators in the same high-dimensional regime as in the previous two chapters, i.e. where L is a fixed integer and where M and N both converge towards infinity in such a way that

$$c_N = \frac{ML}{N} \to c_*, \; 0 < c_* \leq 1.$$

The estimated canonical correlation coefficients coincide with the singular values of matrix $\hat{\mathcal{C}}_N^L = (Y_f Y_f^*)^{-1/2} Y_f Y_p^* (Y_p Y_p^*)^{-1/2}$ because the rows of $(Y_f Y_f^*)^{-1/2} Y_f$ and $(Y_p Y_p^*)^{-1/2} Y_p$ represent orthonormal bases of $\mathcal{Y}_{p,L}$ and $\mathcal{Y}_{f,L}$. In the following, we rather study the singular values to the square, or equivalently the eigenvalues of the $ML \times ML$ matrix $\hat{\mathcal{C}}_N^L \hat{\mathcal{C}}_N^{L*}$ which are also eigenvalues of $N \times N$ matrix $Y_p^*(Y_p Y_p^*)^{-1} Y_p Y_f^* (Y_f Y_f^*)^{-1} Y_f$ up to zeros. Here one can notice that matrices $Y_p^*(Y_p Y_p^*)^{-1} Y_p$ and $Y_f^*(Y_f Y_f^*)^{-1} Y_f$ are projectors and they will be denoted by $\Pi_{p,N}$ and $\Pi_{f,N}$ respectively.
We mention that a number of previous works addressed the behaviour of canonical correlation coefficients in the high-dimensional case. However, the underlying random matrix models are simpler than in the present paper. More specifically, the random matrices $Y_{p,L}$ and $Y_{f,L}$ defined by (5.1, 5.2) are replaced by independent matrices Y_1 and Y_2 with i.i.d. elements, a property that is not verified by $Y_{p,L}$ and $Y_{f,L}$. In 1980, [51] addressed the case of Gaussian i.i.d. entries and derived the limit distribution of the squared canonical correlation coefficients between the row spaces of Y_1 and Y_2. We note that this result appears as a trivial consequence of more recent free probability theory. More recently, [52] extended this result to the case where Y_1 and Y_2 are independent matrices with non Gaussian i.i.d. entries. We also note that [53] took benefit of this result to propose independence tests between 2 sets of i.i.d. high-dimensional samples. We finally mention that [1] extended the result of [51] to the case where Y_1 and Y_2 have Gaussian i.i.d. entries, but this time $\mathbb{E}\{\frac{\mathbf{Y}_1\mathbf{Y}_2^*}{N}\}$ is a non zero low rank matrix.

5.1 With zero signal

This section is dedicated to the case when the signal is absent, so $y_n = v_n$ and Y_p, Y_f coincide with V_p, V_f defined from $(v_n)_{n=1,\ldots,N+2L-1}$. Due to the Gaussianity of the i.i.d. vectors $(v_n)_{n\geq 1}$, it exists i.i.d. $\mathcal{N}_c(0, I_M)$ distributed vectors $(v_{iid,n})_{n\geq 1}$ such that $\mathbb{E}(v_{iid,n} v_{iid,n}^*) = I_M$ verifying $v_n = R_N^{1/2} v_{iid,n}$. It is clear that the row spaces of V_p and V_f coincide with the row spaces of the block Hankel matrices $V_{p,iid}$ and $V_{f,iid}$ defined from vectors $(v_{n,iid})_{n=1,\ldots,N+2L-1}$. Therefore, the correlation coefficients between the 2 pairs of subspaces coincide, and there is no restriction to assume that $R_N = I_M$ in this section.

As before we denote by W_p, W_f the matrices defined by $W_p = \frac{1}{\sqrt{N}} V_p$ and $W_f = \frac{1}{\sqrt{N}} V_f$, then $\Pi_p = W_p^*(W_p W_p^*)^{-1} W_p$ and $\Pi_f = W_f^*(W_f W_f^*)^{-1} W_f$. Also we define the $2ML \times N$ matrix

$$W_N = \begin{pmatrix} W_{p,N} \\ W_{f,N} \end{pmatrix},$$

its elements $(W_{i,j}^m)_{i\leq 2L, j\leq N, m\leq M}$ satisfy

$$\mathbb{E}\{W_{i,j}^m W_{i',j'}^{m'}\} = \delta_{i+j,i'+j'}.$$

where $W_{i,j}^m$ represents the element which lies on the $(m + M(i-1))$-th line and j-th column for $1 \leq m \leq M$, $1 \leq i \leq 2L$ and $1 \leq j \leq N$. For each $j = 1, \ldots, N, \{w_j\}_{j=1}^N, \{w_{p,j}\}_{j=1}^N$ and $\{w_{f,j}\}_{j=1}^N$ are the column of matrices W, W_p and W_f respectively.

It is shown in [32] that the empirical eigenvalue distribution of $W_{i,N} W_{i,N}^*$ for $i = \{p, f\}$ converges towards the Marcenko-Pastur distribution, and that almost surely, for N greater than a random integer, its eigenvalues located in the neighbourhood of $[(1-\sqrt{c_*})^2, (1+\sqrt{c_*})^2]$. Therefore, almost surely, for N large enough, matrices $W_f W_f^*$ and $W_p W_p^*$ are invertible. However, considered as functions of the entries of W_f and W_p, $(W_f W_f^*)^{-1}$ and $(W_p W_p^*)^{-1}$ are not differentiable everywhere. As we use in the following the Nash-Poincaré inequality as well as the integration by parts formula, we introduce a regularization term η_N that avoids any technical problems.

In this chapter we slightly change already introduced notation, that is, we will say that function $f_N(z) = \mathcal{O}_z(\alpha_N)$ if z belongs to a domain $\Omega \subset \mathbb{C}$ and there exist two nice polynomials P_1 and P_2 such that $f_N(z) \leq \alpha_N P_1(|z|)P_2(\frac{1}{\rho(z)})$ for each $z \in \Omega$, where $\rho(z) = \mathrm{dist}(z, \mathbb{R}^+)$. If $\Omega = \mathbb{C} \setminus \mathbb{R}^+$, we will just write $f_N(z) = \mathcal{O}_z(\alpha_N)$ without mentioning the domain, and for any $K \times K$ matrix $A(z)$, by $A(z) = \mathcal{O}_z^K(\alpha_N)$ we mean that each element of $A(z)$ is $\mathcal{O}_z(\alpha_N)$. Finally, we will use a lot the notation $f_N(z) = \mathcal{O}_{z^2}(\alpha_N)$ without mentioning the domain, which will mean that $z^2 \in \mathbb{C}\backslash\mathbb{R}^+$, or equivalently, that $z \in \mathbb{C}\backslash\mathbb{R}$. We notice that if P_1, P_2 and Q_1, Q_2 are nice polynomials, then $P_1(|z|)P_2(\frac{1}{\rho(z)}) + Q_1(|z|)Q_2(\frac{1}{\rho(z)}) \leq (P_1+Q_1)(|z|)(P_2+Q_2)(\frac{1}{\rho(z)})$, from which we conclude that if functions f_1 and f_2 are $\mathcal{O}_z(\alpha)$ then also $f_1(z) + f_2(z) = \mathcal{O}_z(\alpha)$.

5.1.1 Preliminary results

In this subsection we present some useful results concerning our model.

5.1.1.1 Regularization term

This part is dedicated to regularization term and its properties that will help us in further calculations. We define it as

$$\eta_N = \det\,[\phi(W_{f,N}W_{f,N}^*)]\det\,[\phi(W_{p,N}W_{p,N}^*)], \tag{5.3}$$

where ϕ is a smooth function such that

$$\phi(\lambda) = \begin{cases} 1, \text{ for } \lambda \in [(1-\sqrt{c_*})^2 - \epsilon],\ [(1+\sqrt{c_*})^2 + \epsilon], \\ 0, \text{ for } \lambda \in [-\infty,\ (1-\sqrt{c_*})^2 - 2\epsilon] \cup [(1+\sqrt{c_*})^2 + 2\epsilon,\ +\infty] \end{cases}$$

and $\phi(\lambda) \in (0,\,1)$ elsewhere. Taking into account the almost sure behaviour of the eigenvalues of matrices $W_pW_p^*$ and $W_fW_f^*$, $\eta_N = 1$ a.s. for each N larger than a random integer and

$$(W_{i,N}W_{i,N}^*)^{-1}\eta_N \leq \frac{I_{ML}}{(1-\sqrt{c_*})^2 - 2\epsilon}. \tag{5.4}$$

We first mention the following useful property.

Lemma 5.1. *For each $l, k \in \mathbb{N}$ it holds that*

$$\mathbb{E}\{\eta_N^l\} = 1 + \mathcal{O}\left(\frac{1}{N^k}\right) \tag{5.5}$$

Proof. Denote

$$\mathcal{E}_N = \{\text{one of the eigenvalues of } W_pW_p^* \text{ or } W_fW_f^* \text{ escapes from the } [(1-\sqrt{c_*})^2 - \epsilon],\ [(1+\sqrt{c_*})^2 + \epsilon]\}$$

and define another smooth function ϕ_0 as

$$\phi_0(\lambda) = \begin{cases} 0, \text{ for } \lambda \in [(1-\sqrt{c_*})^2],\ [(1+\sqrt{c_*})^2], \\ 1, \text{ for } \lambda \in [-\infty,\ (1-\sqrt{c_*})^2 - \epsilon] \cup [(1+\sqrt{c_*})^2 + \epsilon,\ +\infty] \end{cases}$$

and $\phi_0(\lambda) \in (0,\,1)$ elsewhere. Then we have

$$P(\mathcal{E}_N) \leq P\left(\mathrm{Tr}\phi_0(W_pW_p^*) \geq 1\right) \leq \mathbb{E}\left\{\left(\mathrm{Tr}\phi_0(W_pW_p^*)\right)^{2k}\right\}$$

for all $k \in \mathbb{N}$. In order to evaluate $\mathbb{E}\left\{\left(\mathrm{Tr}\phi_0(W_pW_p^*)\right)^{2k}\right\}$ one can use the same steps as in the proof of Lemma 3.2 [33] and get immediately

$$\mathbb{E}\left\{\left(\mathrm{Tr}\phi_0(W_pW_p^*)\right)^{2k}\right\} = \mathcal{O}\left(\frac{1}{N^{2k}}\right)$$

with $P(\mathcal{E}_N) = \mathcal{O}\left(\frac{1}{N^{2k}}\right)$ for each k. To show (5.5) we write

$$|\mathbb{E}\{\eta_N^l - 1\}|^2 = |\mathbb{E}\{(\eta_N - 1)(1 + \ldots + \eta_N^{l-1})\}|^2 \leq \mathbb{E}\{(\eta_N - 1)^2\}\mathbb{E}\{(1 + \ldots + \eta_N^{l-1})^2\} \leq \kappa\mathbb{E}\{(\eta_N - 1)^2 \mathbf{1}_{\mathcal{E}_N}\}$$

because $\eta_N - 1 = 0$ on $\mathcal{E}_N^c$. Also since by definition $\phi(\lambda) \in [0,1]$ we conclude that $0 \leq \eta_N \leq 1$ and $0 \leq (\eta_N - 1)^2 \leq 1$. This allows us to write that $\kappa\mathbb{E}\{(\eta_N - 1)^2\mathbf{1}_{\mathcal{E}_N}\} \leq \kappa\mathbb{E}\{\mathbf{1}_{\mathcal{E}_N}\} = \kappa P(\mathcal{E}_N) = \mathcal{O}\left(\frac{1}{N^{2k}}\right)$ and so complete the proof. ■

This Lemma permits us to write that $\mathbb{E}\{F\} = \mathbb{E}\{\eta_N^l F\} + \mathcal{O}(N^{-k})$, where F is bounded. Indeed, after applying Schwartz inequality we obtain the familiar term :

$$|\mathbb{E}\{(\eta_N^l - 1)F\}|^2 \leq \mathbb{E}\{(1 - \eta_N^l)^2\}\mathbb{E}\left\{|F|^2\right\} = \kappa(1 - 2(1 + \mathcal{O}\left(\frac{1}{N^k}\right)) + 1 + \mathcal{O}\left(\frac{1}{N^k}\right)) = \mathcal{O}\left(\frac{1}{N^k}\right).$$

Finally, since we use integration by parts formula and Poincaré-Nash inequality, the partial derivatives of η with respect to elements of W_p, W_f will appear and the next lemma is needed.

Lemma 5.2. *Let Ω be the event defined by :*

$$\Omega = \mathcal{E}_N \cap \{\textit{all eigenvalues of } W_p W_p^* \textit{ and } W_f W_f^* \in \mathrm{Supp}(\phi)\}. \tag{5.6}$$

Then it holds that

$$\frac{\partial \eta_N}{\partial W_{i,j}^m} = 0 \textit{ on } \Omega^c \tag{5.7}$$

and

$$\mathbb{E}\left\{\left|\frac{\partial \eta_N}{\partial W_{i,j}^m}\right|^2\right\} = \mathcal{O}\left(\frac{1}{N^k}\right) \tag{5.8}$$

for all $1 \leq m \leq M$, $1 \leq i \leq 2L$, $1 \leq j \leq N$ and each k.

The proof of the lemma is an adaptation of Lemma 11 and calculations from Proposition 4 of [19]

5.1.1.2 Linearisation

From what above we can conclude that for N large enough, $\eta_N \Pi_{i,N} = \Pi_{i,N}$ almost surely and from that, in order to evaluate the almost sure behaviour of the resolvent of $\Pi_{p,N}\Pi_{f,N}$, it is sufficient to study the behaviour of the respective resolvent $Q_N(z)$ defined by

$$Q_N(z) = (\eta_N \Pi_{p,N} \eta_N \Pi_{f,N} - zI)^{-1}$$

As the direct study of $Q_N(z)$ is not obvious, we rather use the well known linearisation trick and introduce the resolvent $\mathbf{Q}_N(z)$ of the $2N \times 2N$ block matrix

$$\mathbf{M}_N = \begin{pmatrix} 0 & \eta_N \Pi_{p,N} \\ \eta_N \Pi_{f,N} & 0 \end{pmatrix}.$$

It is known that $\mathbf{Q}_N(z)$ can be expressed as

$$\mathbf{Q}_N(z) = \begin{pmatrix} zQ_N(z^2) & Q_N(z^2)\eta_N \Pi_{p,N} \\ \eta_N \Pi_{f,N} Q_N(z^2) & z\hat{Q}_N(z^2) \end{pmatrix} \tag{5.9}$$

where $\hat{Q}_N(z)$ is the resolvent of matrix $\eta_N \Pi_{f,N} \eta_N \Pi_{p,N}$. As shown below, it is rather easy to evaluate the asymptotic behaviour of $\mathbf{Q}_N(z)$ using the Poincaré-Nash inequality and the integration by part formula (see Propositions 2.3 and 2.2). Formula (5.9) will then provide all the necessary information on $Q_N(z)$.

Since $Q_N(z)$ and $\mathbf{Q}_N(z)$ are resolvents of non Hermitian matrices, the usual bounds $\|Q_N(z)\| \leq \frac{1}{\mathrm{Im} z}$ and $\|\mathbf{Q}_N(z)\| \leq \frac{1}{\mathrm{Im} z}$ are not necessary valid. Thus a more specific control is needed.

Lemma 5.3. *If* $\text{Im}z \neq 0$ *(i.e.* $z^2 \in \mathbb{C} \setminus \mathbb{R}^+$*), then* $\|\mathbf{Q}(z)\| = \mathcal{O}_{z^2}(1)$.

Proof. In order to bound $\|\mathbf{Q}\|$ it is sufficient to bound each of its blocks : $\mathbf{Q_{pp}}$, $\mathbf{Q_{pf}}$, $\mathbf{Q_{ff}}$ and $\mathbf{Q_{fp}}$. We will start with $\mathbf{Q_{pf}}$. For this we use expression (5.9) for $\mathbf{Q_{pf}}$, the fact that $\Pi_p = \Pi_p^2$ and that $(AB - x)^{-1}A = A(BA - x)^{-1}$ for any matrices A, B. Then,

$$\mathbf{Q_{pf}} = (\eta_N^2 \Pi_p \Pi_f - z^2)^{-1} \eta_N \Pi_p = \eta_N \Pi_p (\eta_N^2 \Pi_p \Pi_f \Pi_p - z^2)^{-1} \Pi_p.$$

Here $(\eta_N^2 \Pi_p \Pi_f \Pi_p - z^2)^{-1}$ is the resolvent of a positive Hermitian matrix, from what follow that its norm can be bounded by $(\rho(z^2))^{-1}$. Since $\|\Pi_p\| \leq 1$ and $\eta_N \leq 1$, we have

$$\|\mathbf{Q_{pf}}\| \leq \frac{1}{\rho(z^2)} \tag{5.10}$$

Analogues for $\mathbf{Q_{fp}}$ we have :

$$\mathbf{Q_{fp}} = \eta_N \Pi_f (\eta_N^2 \Pi_p \Pi_f - z^2)^{-1} = \eta_N \Pi_f (\eta_N^2 \Pi_f \Pi_p \Pi_f - z^2)^{-1} \Pi_f, \tag{5.11}$$

from what we will have the same bound for $\mathbf{Q_{fp}}$. To deal with $\mathbf{Q_{pp}}$ we use again (5.9) and resolvent identity. Thus, we have

$$\mathbf{Q_{pp}} = z(\eta_N^2 \Pi_p \Pi_f - z^2)^{-1} = -\frac{1}{z}(I_N + \eta_N^2 \Pi_p \Pi_f (\eta_N^2 \Pi_p \Pi_f - z^2)^{-1}) = -\frac{1}{z}(I_N + \eta_N \Pi_p \mathbf{Q_{fp}})$$

Obviously $\|I_N + \eta_N \Pi_p \mathbf{Q_{fp}}\| \leq 1 + \frac{1}{\rho(z^2)}$. To show that $|z^{-1}| \leq P(\rho(z^2))$ for some nice polynomial P, we write

$$\frac{1}{|z|^2} \leq \frac{1}{\rho(z^2)} \leq 1 + \frac{1}{\rho(z^2)} \leq \left(1 + \frac{1}{\rho(z^2)}\right)^2 \tag{5.12}$$

This brings us to the conclusion that $\|\mathbf{Q_{pp}}\| = \mathcal{O}_{z^2}(1)$ and so for $\mathbf{Q_{ff}}$. This finishes the proof of the Lemma. ■

Remark 5.1. *It is worth to remark that in the course of the proof we basically get that* $\frac{1}{|z|}\mathcal{O}_{z^2}(1)$ *is still* $\mathcal{O}_{z^2}(1)$ *and since* $|z| \leq \frac{1}{2}(1 + |z|^2)$ *we can also say that* $|z|\mathcal{O}_{z^2}(1) = \mathcal{O}_{z^2}(1)$.

Corollary 5.1. $N^{-1}\text{Tr}\mathbf{Q_{pf}}$ *and* $N^{-1}\text{Tr}\mathbf{Q_{fp}}$ *coincide with the value taken by the Stieltjes transforms evaluated at* z^2 *of some positive measures carried by* $\mathbb{R}^+$*, moreover* $\mathbb{E}\{N^{-1}\text{Tr}\mathbf{Q_{pf}}\}$ *and* $\mathbb{E}\{N^{-1}\text{Tr}\mathbf{Q_{fp}}\}$ *also coincide with the value taken by the Stieltjes transforms evaluated at* z^2 *of some positive measures carried by* $\mathbb{R}^+$ *and of total mass* $c_N + \mathcal{O}(N^{-k})$ *for each* $k \in \mathbb{N}$.

Proof. It is obvious that $N^{-1}\text{Tr}(\eta_N^2 \Pi_p \Pi_f \Pi_p - z)^{-1}$ is the Stieltjes transforms of some positive probability measure carried by $\mathbb{R}^+$ and as a consequence we can easily obtain that function $N^{-1}\text{Tr}\eta_N \Pi_p (\eta_N^2 \Pi_p \Pi_f \Pi_p - z)^{-1} \Pi_p$ is also the Stieltjes transforms of a positive measure carried by $\mathbb{R}^+$ of total mass $N^{-1}\text{Tr}\eta_N \Pi_p^2 = N^{-1}\text{Tr}\eta_N \Pi_p$. But in the Lemma we proved that $\mathbf{Q_{pf}} = \eta_N \Pi_p (\eta_N^2 \Pi_p \Pi_f \Pi_p - z^2)^{-1} \Pi_p$. This gives us immediately the statement of the Lemma. Moreover we can notice that $N^{-1}\mathbb{E}\{\text{Tr}\mathbf{Q_{pf}}\}$ and $N^{-1}\mathbb{E}\{\text{Tr}\mathbf{Q_{fp}}\}$ are also the Stieltjes transforms and the total mass of corresponding measures is $N^{-1}\mathbb{E}\{\text{Tr}\eta_N \Pi_p\} = c_N + \mathcal{O}(N^{-k})$ for any $k \in \mathbb{N}$. ■

Analogous to the two previous chapters, in the following, every $2N \times 2N$ matrix $\mathbf{G}$ will be written as

$$\mathbf{G} = \begin{pmatrix} \mathbf{G_{pp}} & \mathbf{G_{pf}} \\ \mathbf{G_{fp}} & \mathbf{G_{ff}} \end{pmatrix}, \tag{5.13}$$

where the 4 matrices $(\mathbf{G}_{i,j})_{i,j \in p,f}$ are $N \times N$. Sometimes, the blocks will be denoted $\mathbf{G}(pp)$, $\mathbf{G}(pf)$, ...

5.1.1.3 Properties based on the invariance of the complex Gaussian distribution

Lemma 5.4. *The matrix* $\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\}$ *is block diagonal and matrices* $\mathbb{E}\{\eta_N\Pi_i\}$, $\mathbb{E}\{\mathbf{Q_{ij}}\}$, $\mathbb{E}\{\eta_N\Pi_h\mathbf{Q_{ij}}\}$ *and* $\mathbb{E}\{\eta_N\mathbf{Q_{ij}}W_h^*(W_hW_h^*)^{-2}W_h\}$ *are diagonal, for* $i,j,h=\{p,f\}$. *Moreover, if* $i,j,h=\{p,f\}$

$$\mathrm{Tr}\mathbb{E}\{\mathbf{Q_{ij}}\} = \mathrm{Tr}\mathbb{E}\{\mathbf{Q_{\tilde{i}\tilde{j}}}\}, \tag{5.14}$$

$$\mathrm{Tr}\mathbb{E}\{\eta_N\Pi_h\mathbf{Q_{ij}}\} = \mathrm{Tr}\mathbb{E}\{\eta_N\Pi_{\tilde{h}}\mathbf{Q_{\tilde{i}\tilde{j}}}\}, \tag{5.15}$$

where "~" changes index to opposite : $p\to f, f\to p$.

Proof. To prove that all $\mathbb{E}\{\mathbf{Q_{ij}}\}$ are diagonal we consider the new set of vectors $z_k = e^{-ik\theta}y_k$ and construct the matrices Z_p, Z_f in the same way as Y_p and Y_f. It is clear that sequence $(z_n)_{n\in\mathbb{Z}}$ has the same probability distribution that $(y_n)_{n\in\mathbb{Z}}$. Z_p and Z_f can be expressed as

$$Z_p = \begin{pmatrix} e^{-i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta}I_M \end{pmatrix} Y_p \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-(N-1)i\theta} \end{pmatrix},$$

$$Z_f = e^{-Li\theta}\begin{pmatrix} e^{-i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta}I_M \end{pmatrix} Y_f \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-(N-1)i\theta} \end{pmatrix}.$$

Then

$$Z_iZ_i^* = \begin{pmatrix} e^{-i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta}I_M \end{pmatrix} Y_iY_i^* \begin{pmatrix} e^{i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{Li\theta}I_M \end{pmatrix}, \tag{5.16}$$

$$(Z_iZ_i^*)^{-1} = \begin{pmatrix} e^{-i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-Li\theta}I_M \end{pmatrix} (Y_iY_i^*)^{-1} \begin{pmatrix} e^{i\theta}I_M & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{Li\theta}I_M \end{pmatrix} \tag{5.17}$$

so for the corresponding functions $\phi(Z_fZ_f^*) = \phi(Y_fY_f^*)$ and $\phi(Z_pZ_p^*) = \phi(Y_pY_p^*)$. This imply that new regularization term $\eta^z = \det\phi(Z_pZ_p^*)\det\phi(Z_fZ_f^*)$ will remain the same, i.e. $\eta^z=\eta$. Next we define $\Pi_i^z = Z_i^*(Z_iZ_i^*)^{-1}Z_i$, $i=\{p,f\}$ it holds that

$$\Pi_i^z = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{(N-1)i\theta} \end{pmatrix} \Pi_i \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-(N-1)i\theta} \end{pmatrix} \tag{5.18}$$

for $i=\{p,f\}$. Similarly to $\mathbf{Q}$ we define matrix $\mathbf{Q^Z} = \begin{pmatrix} -zI_{ML} & \eta^z\Pi_p^z \\ \eta^z\Pi_f^z & -zI_{ML} \end{pmatrix}^{-1}$ and obtain immediately that

$$\mathbb{E}\{\mathbf{Q^Z}\} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mathbb{E}\{\mathbf{Q}\} \begin{pmatrix} A^* & 0 \\ 0 & A^* \end{pmatrix},$$

where $N\times N$ matrix A defined as

$$A = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{(N-1)i\theta} \end{pmatrix}$$

Obviously for each $N\times N$ block $\mathbb{E}\{\mathbf{Q_{ij}}^z\}$, $i,j=\{p,f\}$, we have

$$\mathbb{E}\{\mathbf{Q_{ij}}^z\} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{(N-1)i\theta} \end{pmatrix} \mathbb{E}\{\mathbf{Q_{ij}}\} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-(N-1)i\theta} \end{pmatrix}$$

and as consequence

$$\mathbb{E}\{\eta_N \Pi_h^z {\mathbf{Q_{ij}}}^z\} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{(N-1)i\theta} \end{pmatrix} \mathbb{E}\{\eta_N \Pi_h \mathbf{Q_{ij}}\} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-(N-1)i\theta} \end{pmatrix},$$

for $h = \{p, f\}$. Since $\mathbb{E}\{\mathbf{Q}^{\mathbf{Z}}\} = \mathbb{E}\{\mathbf{Q}\}$, for every element $\mathbb{E}\{{\mathbf{Q_{ij}}}^{k,l}\}$ with $1 \leq k, l \leq N$ and $i, j, h = \{p, f\}$ it holds

$$\begin{aligned} &\mathbb{E}\{{\mathbf{Q_{ij}}}^{k,l}\} = e^{(k-1)i\theta}\mathbb{E}\{{\mathbf{Q_{ij}}}^{k,l}\}e^{-(l-1)i\theta} = e^{(k-l)i\theta}\mathbb{E}\{{\mathbf{Q_{ij}}}^{k,l}\} \\ &\mathbb{E}\{\eta_N(\Pi_h\mathbf{Q_{ij}})^{k,l}\} = e^{(k-1)i\theta}\mathbb{E}\{\eta_N(\Pi_h\mathbf{Q_{ij}})^{k,l}\}e^{-(l-1)i\theta} = e^{(k-l)i\theta}\mathbb{E}\{\eta_N(\Pi_h\mathbf{Q_{ij}})^{k,l}\} \end{aligned}$$

This proves that $\mathbb{E}\{{\mathbf{Q_{ij}}}^{k,l}\} = 0$ and $\mathbb{E}\{\eta_N(\Pi_h\mathbf{Q_{ij}})^{k,l}\} = 0$ if $k \neq l$, as expected. Analogous we can prove the same results for $\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\}$, $\mathbb{E}\{\eta_N\Pi_i\}$ and $\mathbb{E}\{\eta_N\mathbf{Q_{ij}}W_h^*(W_hW_h^*)^{-2}W_h\}$ from (5.17) and (5.18).

To prove (5.14) let us consider sequence z defined by $z_n = y_{-n+N+2L}$ for each n. Again, the distribution of z_n will remain the same and it is easy to see that for $i \in \{p, f\}$ Z_i can be found as

$$Z_i = \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix} Y_{\tilde{i}} \begin{pmatrix} 0 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix},$$

and as consequence

$$Z_iZ_i^* = \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix} Y_{\tilde{i}}Y_{\tilde{i}}^* \begin{pmatrix} 0 & \dots & I_M \\ \vdots & & \vdots \\ I_M & \dots & 0 \end{pmatrix}$$

Here one can see that $Z_iZ_i^*$ is a unitary transformation of $Y_{\tilde{i}}Y_{\tilde{i}}^*$, so both matrices has the same eigenvalues, which means that $\phi(Z_iZ_i^*) = \phi(Y_{\tilde{i}}Y_{\tilde{i}}^*)$. This imply that new regularization term $\eta^z = \det\phi(Z_pZ_p^*)\det\phi(Z_fZ_f^*)$ will remain the same, i.e. $\eta^z = \eta$. Next we find corresponding expressions for $\Pi_{p,f}^z$ and $\mathbf{Q}^{\mathbf{z}}$. It is easy to see that $\Pi_p^z = A\Pi_fA$ and $\Pi_f^z = A\Pi_pA$, where this time $A = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix}$ is a $N \times N$ matrix. From this, we obtain that

$$\mathbb{E}\{\mathbf{Q}^{\mathbf{Z}}\} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mathbb{E}\Big\{ \begin{pmatrix} -zI_N & \eta\Pi_f \\ \eta\Pi_p & -zI_N \end{pmatrix}^{-1} \Big\} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Using the formula for inverse block matrices and fact that $\mathbb{E}\{\mathbf{Q}^{\mathbf{Z}}\} = \mathbb{E}\{\mathbf{Q}\}$, we obtain that $\mathbb{E}\{\mathbf{Q_{pp}}\} = A\mathbb{E}\{\mathbf{Q_{ff}}\}A$ and $\mathbb{E}\{\mathbf{Q_{pf}}\} = A\mathbb{E}\{\mathbf{Q_{fp}}\}A$. This immediately implies that for every $1 \leq k \leq N$ and $h, i, j = \{p, f\}$ we have $\mathbb{E}\{(\mathbf{Q_{ij}})^{k,k}\} = \mathbb{E}\{(\mathbf{Q_{\tilde{i}\tilde{j}}})^{N+1-k,N+1-k}\}$ and $\mathbb{E}\{\eta_N(\Pi_h\mathbf{Q_{ij}})^{k,k}\} = \mathbb{E}\{\eta_N(\Pi_{\tilde{h}}\mathbf{Q_{\tilde{i}\tilde{j}}})^{N+1-k,N+1-k}\}$. As consequence $\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{ij}}\} = \mathbb{E}\{\mathrm{Tr}\mathbf{Q_{\tilde{i}\tilde{j}}}\}$ and $\mathbb{E}\{\eta_N\Pi_h\mathbf{Q_{ij}}\} = \mathbb{E}\{\eta_N\Pi_{\tilde{h}}\mathbf{Q_{\tilde{i}\tilde{j}}}\}$ as expected. ■

Previous Lemma gives us that matrices $\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\}$ and $\mathbb{E}\{\eta_N\Pi_i\}$ are diagonal, in the next Lemma we will prove that they are actually a multiple of identity matrix up to an error term.

Lemma 5.5. *For $i = \{p, f\}$, we have :*

$$\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\} = \frac{1}{1-c_N}I_{ML} + \mathcal{O}^{ML}\left(\frac{1}{N^{3/2}}\right) \tag{5.19}$$

$$\mathbb{E}\{\eta_N\Pi_i\} = c_NI_N + \mathcal{O}^N\left(\frac{1}{N^{3/2}}\right). \tag{5.20}$$

Moreover, $(ML)^{-1}\mathrm{Tr}\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\} = (1-c_N)^{-1} + \mathcal{O}(\frac{1}{N^2})$.

Proof. We consider $i = p$, obviously for $i = f$ the proof is analogous. In the following we drop index i and denote $G = (WW^*)^{-1}$. To prove this lemma we will use the integration by parts formula (Proposition 2.2) for $\eta_N G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \bar{W}^{m_3}_{j_1,i_3}$:

$$\mathbb{E}\{\eta_N G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \bar{W}^{m_3}_{j_1,i_3}\} = \sum_{m',i',j'} \mathbb{E}\{\bar{W}^{m_3}_{j_1,i_3} W^{m'}_{i',j'}\}$$
$$\times \left(\mathbb{E}\left\{ \frac{\partial \eta_N}{\partial W^{m'}_{i',j'}} G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \right\} + \mathbb{E}\left\{ \eta_N \frac{\partial G^{m_1m_2}_{i_1i_2}}{\partial W^{m'}_{i',j'}} W^{m_2}_{i_2,j_2} \right\} + \mathbb{E}\left\{ \eta_N G^{m_1m_2}_{i_1i_2} \frac{\partial W^{m_2}_{i_2,j_2}}{\partial W^{m'}_{i',j'}} \right\} \right) \quad (5.21)$$

Lemma 5.2 implies that the first term of r.h.s of (5.21) is of order $\mathcal{O}(N^{-k})$ for each k. Indeed,

$$\mathbb{E}\left\{ \frac{\partial \eta_N}{\partial W^{m'}_{i',j'}} G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \right\} = \mathbb{E}\left\{ \mathbf{1}_\Omega \frac{\partial \eta_N}{\partial W^{m'}_{i',j'}} G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \right\}$$

and Schwartz inequality leads to

$$\left| \mathbb{E}\left\{ \mathbf{1}_\Omega \frac{\partial \eta_N}{\partial W^{m'}_{i',j'}} G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \right\} \right|^2 \le \mathbb{E}\left\{ \left| \frac{\partial \eta_N}{\partial W^{m'}_{i',j'}} \right|^2 \right\} \mathbb{E}\left\{ \left| \mathbf{1}_\Omega G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \right|^2 \right\} \quad (5.22)$$

On event Ω all eigenvalues of WW^* belong to $((1-\sqrt{c_*})^2 - 2\epsilon, (1+\sqrt{c_*})^2 + 2\epsilon)$, so $\|G\mathbf{1}_\Omega\|$ and $\|W\mathbf{1}_\Omega\|$ are bounded, therefore we have immediately that $\left|\mathbf{1}_\Omega G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2}\right|$ is bounded by some nice constant. Then, after some calculations (5.21) becomes

$$\mathbb{E}\{\eta_N G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \bar{W}^{m_3}_{j_1,i_3}\} = \frac{1}{N} \sum_{m',i',j'} \delta_{m',m_3} \delta_{i_3+j_1,i'+j'}$$
$$\times \left(-\mathbb{E}\left\{ \eta_N G^{m_1m'}_{i_1i'} (W^*G)^{m_2}_{j',i_2} W^{m_2}_{i_2,j_2} \right\} + \mathbb{E}\{\eta_N G^{m_1m_2}_{i_1i_2} \delta_{m',m_2} \delta_{i_2,i'} \delta_{j_2,j'}\} \right) + \mathcal{O}\left(\frac{1}{N^k}\right) \quad (5.23)$$

Defining $l = i_3 - i' = j' - j_1$ which changes from $-L+1$ to $L-1$ and taking into account (2.3) we get $\delta_{m',m_3}\delta_{i_3+j_1,i'+j'} = (J^{(l)}_L \otimes I_M)^{m'm_3}_{i'i_3} (J^{(l)}_N)_{j_1j'}$. Then, after summing over i', j' and m', (5.23) becomes

$$\mathbb{E}\{\eta_N G^{m_1m_2}_{i_1i_2} W^{m_2}_{i_2,j_2} \bar{W}^{m_3}_{j_1,i_3}\} = -\frac{1}{N} \mathbb{E}\left\{ \eta_N \left(G(J^{(l)}_L \otimes I_M) \right)^{m_1m_3}_{i_1i_3} (J^{(l)}_N W^* G)^{m_2}_{j_1,i_2} W^{m_2}_{i_2,j_2} \right\}$$
$$+ \frac{1}{N} \mathbb{E}\{\eta_N G^{m_1m_2}_{i_1i_2} (J^{(l)}_L \otimes I_M)^{m_2m_3}_{i_2i_3} (J^{(l)}_N)_{j_1j_2}\} + \mathcal{O}\left(\frac{1}{N^k}\right)$$

and again, this time summing both sides over i_2, m_2 :

$$\mathbb{E}\{\eta_N (GW)^{m_1}_{i_1j_2} \bar{W}^{m_3}_{j_1,i_3}\} = -\frac{1}{N} \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \eta_N (G(J^{(l)}_L \otimes I_M))^{m_1m_3}_{i_1i_3} (J^{(l)}_N \Pi)_{j_1j_2} \right\}$$
$$+ \frac{1}{N} \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \eta_N (G(J^{(l)}_L \otimes I_M))^{m_1m_3}_{i_1i_3} (J^{(l)}_N)_{j_1j_2} \right\} + \mathcal{O}\left(\frac{1}{N^k}\right). \quad (5.24)$$

At this point in order to prove (5.19) we take $j_2 = j_1$ and sum over this index, then, since $GWW^* = I_{ML}$ we have

$$\mathbb{E}\{\eta_N\} I_{ML} = -\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \eta_N G(J^{(l)}_L \otimes I_M) \frac{1}{N} \mathrm{Tr}(J^{(l)}_N \Pi) \right\} + \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \eta_N G(J^{(l)}_L \otimes I_M) \frac{1}{N} \mathrm{Tr} J^{(l)}_N \right\} + \mathcal{O}\left(\frac{1}{N^k}\right)$$

Obviously $\frac{1}{N}\mathrm{Tr}J_N^{(l)}$ is equal to 0 for $l \neq 0$ and to 1 if $l = 0$ and as was discussed above we can replace $\mathbb{E}\{\eta_N\}$ by 1 on the l.h.s. while adding term $\mathcal{O}(N^{-k})$ and η_N by η_N^2 on the r.h.s. Then

$$I_{ML} = -\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{\eta_N G(J_L^{(l)} \otimes I_M)\right\} \mathbb{E}\left\{\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)\right\} - \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{\eta_N G(J_L^{(l)} \otimes I_M)\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)^{\circ}\right\} + \mathbb{E}\left\{\eta_N G\right\} + \mathcal{O}\left(\frac{1}{N^k}\right) \quad (5.25)$$

Lemma 5.4 implies that $\mathbb{E}\{\eta_N\Pi\}$ is diagonal, so $\mathbb{E}\left\{\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)\right\} = 0$ for all $l \neq 0$ and moreover since $\frac{1}{N}\mathrm{Tr}\Pi = c_N$ it is easy to see that $\mathbb{E}\left\{\frac{1}{N}\mathrm{Tr}(\eta_N\Pi)\right\} = c_N + \mathcal{O}(N^{-k})$ for each k. Then from the last equation we derive immediately the expression for $\mathbb{E}\{\eta_N G\}$:

$$\mathbb{E}\left\{\eta_N(WW^*)^{-1}\right\} = \frac{1}{1-c}I_{ML} + \frac{1}{1-c}\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{\eta_N G(J_L^{(l)} \otimes I_M)\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)^{\circ}\right\} + \mathcal{O}\left(\frac{1}{N^k}\right) \quad (5.26)$$

Finally, we show that each element of matrix $\sum \mathbb{E}\left\{\eta_N G(J_L^{(l)} \otimes I_M)\frac{1}{N}\mathrm{Tr}(J_N^{(l)}(\eta_N\Pi)^{\circ})\right\}$ is of order $\mathcal{O}(N^{-3/2})$, for this we apply Schwartz inequality :

$$\left|\mathbb{E}\left\{(\mathbf{f}_{i_1}^{m_1})^*\eta_N G(J_L^{(l)} \otimes I_M)\mathbf{f}_{i_2}^{m_2}\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi^{\circ})\right\}\right| \leq \left(\mathbf{Var}\left((\mathbf{f}_{i_1}^{m_1})^*\eta_N G(J_L^{(l)} \otimes I_M)\mathbf{f}_{i_2}^{m_2}\right)\mathbf{Var}\left(\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)\right)\right)^{1/2}$$

In order to evaluate variances one should follow the steps of the proof of Proposition 3.1 [32]. In [32], matrix ηG is replaced by the resolvent of WW^* evaluated at $z \in \mathbb{C}^+$. The proof of Proposition 3.1 [32] uses the fact that the norm of this resolvent is bounded by $\frac{1}{\mathrm{Im}z}$, a result that is of course not true in the present context. However, the above upper bound is replaced by $\eta_N G \leq \kappa I_N$ (see (5.4)). This allows to obtain the same estimations as in Proposition 3.1 [32] :

$$\mathbf{Var}\left((\mathbf{f}_{i_1}^{m_1})^*\eta_N G(J_L^{(l)} \otimes I_M)\mathbf{f}_{i_2}^{m_2}\right) = \mathcal{O}\left(\frac{1}{N}\right)$$
$$\mathbf{Var}\left(\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)\right) = \mathcal{O}\left(\frac{1}{N^2}\right)$$
$$\mathbf{Var}\left(\frac{1}{ML}\mathrm{Tr}\eta_N G(J_L^{(l)} \otimes I_M)\right) = \mathcal{O}\left(\frac{1}{N^2}\right)$$

and conclude (5.19).
To estimate the expectation of $(ML)^{-1}\mathrm{Tr}\eta_N(WW^*)^{-1}$ we take a normalized trace from both sides of (5.26) and use again the Schwartz inequality for an error term :

$$\left|\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}(\eta_N G(J_L^{(l)} \otimes I_M))\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi^{\circ})\right\}\right| \leq \left(\mathbf{Var}\left(\frac{1}{ML}\mathrm{Tr}\eta_N G(J_L^{(l)} \otimes I_M)\right)\mathbf{Var}\left(\frac{1}{N}\mathrm{Tr}(\eta_N J_N^{(l)}\Pi)\right)\right)^{1/2} = \mathcal{O}\left(\frac{1}{N^2}\right)$$

Then we get immediately $(ML)^{-1}\mathrm{Tr}\mathbb{E}\{\eta_N(W_iW_i^*)^{-1}\} = (1-c_N)^{-1} + \mathcal{O}(\frac{1}{N^2})$.
Finally, to prove (5.20) we return to equation (5.24) but this time we take $m_1 = m_3$, $i_1 = i_3$ and sum both sides over these indexes :

$$\mathbb{E}\{\eta_N\Pi\} = -c_N\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}(\eta_N G(J_L^{(l)} \otimes I_M))(J_N^{(l)}\Pi)\right\} + c_N\sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}(\eta_N G(J_L^{(l)} \otimes I_M))J_N^{(l)}\right\} + \mathcal{O}\left(\frac{1}{N^k}\right)$$

Analogous to what we have seen above, we replace η_N by η_N^2 in the first term of r.h.s. and remark that $\mathbb{E}\{\mathrm{Tr}(\eta_N G(J_L^{(l)}\otimes I_M))\}=0$ for all $l\neq 0$, since $\mathbb{E}\{\eta_N G\}$ is block diagonal, moreover $\mathbb{E}\{(ML)^{-1}\mathrm{Tr}(\eta_N G)\}=(1-c_N)^{-1}+\mathcal{O}(\frac{1}{N^2})$, then after trivial algebra we get

$$\mathbb{E}\{\eta_N\Pi\}=c_N I_N+\mathcal{O}\left(\frac{1}{N^2}\right)+\sum_{l=-(L-1)}^{L-1}\mathbb{E}\left\{\frac{1}{ML}\mathrm{Tr}(\eta_N G(J_L^{(l)}\otimes I_M))^{\circ}\eta_N J_N^{(l)}\Pi\right\}$$

Like in previous case, with Schwartz inequality we obtain the necessary error terms. ■

5.1.2 Expression of matrix $\mathbb{E}\{\mathbf{Q}\}$ obtained using the integration by parts formula

Now we return to the expression of $\mathbf{Q}(z)$. Using the resolvent identity we have

$$z\mathbf{Q}(z)=-I_{2N}+\mathbf{Q}(z)\begin{pmatrix}0 & \eta\Pi_p\\ \eta\Pi_f & 0\end{pmatrix}=-I_{2N}+\begin{pmatrix}\mathbf{Q_{pf}}(z)\eta\Pi_f & \mathbf{Q_{pp}}(z)\eta\Pi_p\\ \mathbf{Q_{ff}}(z)\eta\Pi_f & \mathbf{Q_{fp}}(z)\eta\Pi_p\end{pmatrix}. \tag{5.27}$$

The goal will be to express all four blocks of r.h.s. in terms of $\mathbf{Q}(z)$ with help of integration by parts formula, Proposition 2.2. We start with $\mathbf{Q_{pp}}(z)\eta\Pi_p$.

$$\mathbb{E}\{(\mathbf{Q_{pp}}\eta\Pi_p)_{rs}\}=\sum_{t=1}^{N}\sum_{i_1,i_2=1}^{L}\sum_{m_1,m_2=1}^{M}\mathbb{E}\Big\{\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\bar{W}_{p,i_1t}^{m_1}\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}W_{p,i_2s}^{m_2}\Big\}=\sum\mathbb{E}\{\bar{W}_{p,i_1t}^{m_1}W_{i_3u}^{m_3}\}$$
$$\times\mathbb{E}\left\{\frac{\partial\left(\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}W_{p,i_2s}^{m_2}\right)}{\partial W_{i_3u}^{m_3}}\right\}=\frac{1}{N}\sum\mathbb{E}\left\{\delta_{m_1,m_3}\delta_{i_1+t,i_3+u}\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}\frac{\partial W_{p,i_2s}^{m_2}}{\partial W_{i_3u}^{m_3}}\right.$$
$$\left.+\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\frac{\partial\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}}{\partial W_{i_3u}^{m_3}}W_{p,i_2s}^{m_2}+\frac{\partial\mathbf{Q}_{\mathbf{pp}}^{rt}}{\partial W_{i_3u}^{m_3}}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}W_{p,i_2s}^{m_2}+\mathbf{Q}_{\mathbf{pp}}^{rt}\frac{\partial\eta}{\partial W_{i_3u}^{m_3}}\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}W_{p,i_2s}^{m_2}\right\} \tag{5.28}$$

Here we take derivative with respect to each element of W, so index i_3 takes values from 1 to $2L$. We are going to denote each term of r.h.s. without expectation by T_1, T_2, T_3, T_4 respectively and treat them separately. First one is obvious

$$T_1=\frac{1}{N}\sum\delta_{m_1,m_3}\delta_{i_1+t,i_3+u}\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}\frac{\partial W_{p,i_2s}^{m_2}}{\partial W_{i_3u}^{m_3}}=\frac{1}{N}\sum\delta_{m_1,m_2}\delta_{i_1+t,i_2+s}\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}$$

We define a new index $l=i_1-i_2$ which obviously takes values from $\{-(L-1,\dots,L-1)\}$, then we can rewrite $\delta_{i_1+t,i_2+s}=\delta_{i_1-i_2,l}\delta_{s-t,l}=(J_M^{(l)})_{i_2i_1}(J_N^{(l)})_{ts}$ and taking into account (2.3) we obtain

$$T_1=\frac{1}{N}\sum(J_N^{(l)})_{ts}(J_L^{(l)}\otimes I_M)_{i_2i_1}^{m_2m_1}\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_2}^{m_1m_2}=\sum_{l=-(L-1)}^{L-1}\left(\mathbf{Q_{pp}}J_N^{(l)}\right)_{rs}\frac{1}{N}\mathrm{Tr}\left((J_L^{(l)}\otimes I_M)\eta(W_pW_p^*)^{-1}\right) \tag{5.29}$$

Now we take an expectation and rewrite

$$\mathbb{E}\{T_1\}=\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\left(\mathbf{Q_{pp}}J_N^{(l)}\right)_{rs}\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left((J_L^{(l)}\otimes I_M)\eta(W_pW_p^*)^{-1}\right)\Big\}$$
$$+\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\left(\mathbf{Q}_{\mathbf{pp}}^{\circ}J_N^{(l)}\right)_{rs}\frac{1}{N}\mathrm{Tr}\left((J_L^{(l)}\otimes I_M)\eta(W_pW_p^*)^{-1}\right)\Big\}$$

In the obtained equation we denote the second term of r.h.s by $T_1^{\mathcal{E}}$. According to (5.19) $\mathbb{E}\{(ML)^{-1}\mathrm{Tr}\eta(W_pW_p^*)^{-1}\}=\frac{1}{1-c_N}+\mathcal{O}(\frac{1}{N^2})$, it means that if $l=0$ we have

$$\frac{1}{N}\mathbb{E}\{\mathrm{Tr}(\eta(W_pW_p^*)^{-1})\}=\frac{c_N}{(1-c_N)}+\mathcal{O}\left(\frac{1}{N^2}\right)$$

and if $l \neq 0$ then from Lemma 5.4 we have $\frac{1}{N}\mathbb{E}\Big\{ \mathrm{Tr}\left((J_L^{(l)} \otimes I_M)\eta(W_pW_p^*)^{-1}\right) \Big\} = 0$. Since resolvent is bounded (see Lemma 5.3), the only term which gives impact appears when $l = 0$ and

$$\mathbb{E}\{T_1\} = \frac{c_N}{1-c_N}\mathbb{E}\Big\{ (\mathbf{Q_{pp}})_{rs} \Big\} + \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right) + T_1^{\mathcal{E}}. \tag{5.30}$$

For second term we have

$$T_2 = -\frac{1}{N}\sum \delta_{m_1,m_3}\delta_{i_1+t,i_3+u}\mathbf{Q}_{\mathbf{pp}}^{rt}\eta\left((W_pW_p^*)^{-1}\right)_{i_1i_3}^{m_1m_3}\left(W_p^*(W_pW_p^*)^{-1}\right)_{ui_2}^{m_2}W_{p,i_2s}^{m_2}$$

Here we take $l = i_1 - i_3$ and again $-(L-1) \leq l \leq L-1$, then $\delta_{i_1+t,i_3+u} = \delta_{i_1-i_3,l}\delta_{u-t,l} = (J_M^{(l)})_{i_3i_1}(J_N^{(l)})_{tu}$. This gives us

$$T_2 = -\sum_{l=-(L-1)}^{L-1}\left(\eta\mathbf{Q_{pp}}J_N^{(l)}\Pi_p\right)_{rs}\frac{1}{N}\mathrm{Tr}\left((J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}\right) \tag{5.31}$$

Taking the expectation and replacing η by η^2, we have

$$\begin{aligned}\mathbb{E}\{T_2\} = &-\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\left(\eta\mathbf{Q_{pp}}J_N^{(l)}\Pi_p\right)_{rs}\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}\right)\Big\}\\ &-\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\left(\eta\mathbf{Q_{pp}}J_N^{(l)}\Pi_p\right)_{rs}^{\circ}\frac{1}{N}\mathrm{Tr}\left(\eta(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}\right)\Big\} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right)\end{aligned}$$

Analogues to previous case, in the last equation we denote the second term of r.h.s. by $T_2^{\mathcal{E}}$ and notice that in the first term, according to Lemma 5.4, all terms except of when $l = 0$ are zeros, and $\mathbb{E}\{(ML)^{-1}\mathrm{Tr}\eta(W_pW_p^*)^{-1}\} = \frac{1}{1-c_N} + \mathcal{O}(\frac{1}{N^2})$, then

$$\mathbb{E}\{T_2\} = -\frac{c_N}{1-c_N}\mathbb{E}\Big\{(\eta\mathbf{Q_{pp}}\Pi_p)_{rs}\Big\} + T_2^{\mathcal{E}} + \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right) \tag{5.32}$$

To deal with third term, T_3, we first should find the derivatives of resolvent. For this we write

$$\partial\mathbf{Q} = -\mathbf{Q}\partial\begin{pmatrix}0 & \eta\Pi_p\\ \eta\Pi_f & 0\end{pmatrix}\mathbf{Q} = -\begin{pmatrix}\mathbf{Q_{pf}}\partial(\eta\Pi_f)\mathbf{Q_{pp}} + \mathbf{Q_{pp}}\partial(\eta\Pi_p)\mathbf{Q_{fp}} & \mathbf{Q_{pf}}\partial(\eta\Pi_f)\mathbf{Q_{pf}} + \mathbf{Q_{pp}}\partial(\eta\Pi_p)\mathbf{Q_{ff}}\\ \mathbf{Q_{ff}}\partial(\eta\Pi_f)\mathbf{Q_{pp}} + \mathbf{Q_{fp}}\partial(\eta\Pi_p)\mathbf{Q_{fp}} & \mathbf{Q_{ff}}\partial(\eta\Pi_f)\mathbf{Q_{pf}} + \mathbf{Q_{fp}}\partial(\eta\Pi_p)\mathbf{Q_{ff}}\end{pmatrix} \tag{5.33}$$

Now we take a derivative with respect to the element $W_{i_3u}^{m_3}$. As was discussed before, since $\|\mathbf{Q}\|$ and $\|\Pi_{p,f}\|$ are bounded (see Lemma 5.3), the expectation of the terms that correspond to $\frac{\partial\eta}{\partial W_{i_3u}^{m_3}}$ can be bounded by any power of N^{-1}. This justifies that we can put all this terms together and denote result general matrix by $\mathcal{E}$ for which $\mathbb{E}\{\mathcal{E}\} = \mathcal{O}_{z^2}(N^{-k})$ for any k, more precisely it will be discussed in Section 5.1.4. Finally we recall the classic formula for the derivative of projector Π_p (for Π_f the formula is analogues)

$$\delta\Pi_p = \Pi_p^{\perp}\delta(W_p^*W_p)(W_p^*W_p)^{\#} + (W_p^*W_p)^{\#}\delta(W_p^*W_p)\Pi_p^{\perp} \tag{5.34}$$

where $(W_p^*W_p)^{\#}$ is the pseudoinverse of $W_p^*W_p$ and in this case is equal to $W_p^*(W_pW_p^*)^{-2}W_p$. Since we are taking the derivative with respect to the $W_{i_3u}^{m_3}$, formula (5.34) can be simplified, more precisely :

$$\frac{\partial\Pi_p}{\partial W_{i_3u}^{m_3}} = \left(\Pi_p^{\perp}W_p^*\mathbf{f}_{i_3}^{m_3}\mathbf{e}_uW_p^*(W_pW_p^*)^{-2}W_p + W_p^*(W_pW_p^*)^{-2}W_pW_p^*\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u\Pi_p^{\perp}\right)\mathbf{1}_{i_3\leq L}.$$

Here $\mathbf{f}_{i_3}^{m_3}$ is the basis vector of $\mathbb{R}^{ML}$, obviously if $i_3 > L$ the derivative is 0. Also since $\Pi_p^{\perp}W_p^* = 0$ the first term is disappear and finally we have

$$\frac{\partial\Pi_p}{\partial W_{i_3u}^{m_3}} = W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u^*\Pi_p^{\perp}\mathbf{1}_{i_3\leq L}$$

For Π_f the formula is absolutely analogous, but instead of $\mathbf{f}_{i_3}^{m_3}$ we will have $\mathbf{f}_{i_3-L}^{m_3}$:

$$\frac{\partial \Pi_f}{\partial W_{i_3u}^{m_3}} = W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_3-L}^{m_3}\mathbf{e}_u^*\Pi_f^\perp \mathbf{1}_{i_3>L}$$

Putting these expression in (5.33) we have

$$\begin{aligned}\frac{\partial \mathbf{Q}}{\partial W_{i_3u}^{m_3}} = &-\eta \mathbf{1}_{i_3\le L}\begin{pmatrix}\mathbf{Q_{pp}}(W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u^*\Pi_p^\perp \mathbf{Q_{fp}} & \mathbf{Q_{pp}}(W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u^*\Pi_p^\perp \mathbf{Q_{ff}} \\ \mathbf{Q_{fp}}(W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u^*\Pi_p^\perp \mathbf{Q_{fp}} & \mathbf{Q_{fp}}(W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_3}^{m_3}\mathbf{e}_u^*\Pi_p^\perp \mathbf{Q_{ff}}\end{pmatrix} \\ &-\eta \mathbf{1}_{i_3>L}\begin{pmatrix}\mathbf{Q_{pf}}(W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_3-L}^{m_3}\mathbf{e}_u^*\Pi_f^\perp \mathbf{Q_{pp}} & \mathbf{Q_{pf}}(W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_3-L}^{m_3}\mathbf{e}_u^*\Pi_f^\perp \mathbf{Q_{pf}} \\ \mathbf{Q_{ff}}(W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_3-L}^{m_3}\mathbf{e}_u^*\Pi_f^\perp \mathbf{Q_{pp}} & \mathbf{Q_{ff}}(W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_3-L}^{m_3}\mathbf{e}_u^*\Pi_f^\perp \mathbf{Q_{pf}}\end{pmatrix} + \mathcal{E}\end{aligned} \quad (5.35)$$

Now we are ready to deal with term T_3, firs we sum over i_2, m_2 :

$$\begin{aligned}T_3 = &-\frac{1}{N}\sum \delta_{m_1,m_3}\delta_{i_1+t,i_3+u}\eta^2\left(\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}\right)_{ri_3}^{m_3}\left(\Pi_p^\perp \mathbf{Q_{fp}}\right)_{ut}\left((W_pW_p^*)^{-1}W_p\right)_{i_1s}^{m_1}\mathbf{1}_{i_3\le L} \\ &-\frac{1}{N}\sum \delta_{m_1,m_3}\delta_{i_1+t,i_3+u}\eta^2\left(\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1}\right)_{ri_3-L}^{m_3}\left(\Pi_f^\perp \mathbf{Q_{pp}}\right)_{ut}\left((W_pW_p^*)^{-1}W_p\right)_{i_1s}^{m_1}\mathbf{1}_{i_3>L} + \mathcal{E}\end{aligned} \quad (5.36)$$

Then for first term of obtained r.h.s. we again define $l = i_1 - i_3$, since $i_3 \le L$ index $l \in \{-(L-1), \dots, L-1\}$ and $\delta_{i_1+t,i_3+u} = \delta_{i_1-i_3,l}\delta_{u-t,l} = (J_L^{(l)})_{i_3i_1}(J_N^{(l)})_{tu}$. In the second term we first change the variable i_3 $i_3 \to i_3+L$, then new i_3 runs from 1 to L and the term itself becomes

$$\frac{1}{N}\sum \delta_{m_1,m_3}\delta_{i_1+t,i_3+L+u}\eta^2(\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1})_{ri_3}^{m_3}(\Pi_f^\perp \mathbf{Q_{pp}})_{ut}((W_pW_p^*)^{-1}W_p)_{i_1s}^{m_1}\mathbf{1}_{i_3<L}.$$

Now as just above we denote $l = i_1 - i_3$, then $\delta_{i_1+t,i_3+L+u} = \delta_{i_1-i_3,l}\delta_{u-t,l-L} = (J_L^{(l)})_{i_3,i_1}(J_N^{(l-L)})_{tu}$. That gives us, after summing over i_3, j_3, m_3 and t, u :

$$\begin{aligned}T_3 = &-\sum_{l=-(L-1)}^{L-1}\eta^2\left(\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}\frac{1}{N}\mathrm{Tr}\left(\Pi_p^\perp \mathbf{Q_{fp}}J_N^{(l)}\right) \\ &-\sum_{l=-(L-1)}^{L-1}\eta^2\left(\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}\frac{1}{N}\mathrm{Tr}\left(\Pi_f^\perp \mathbf{Q_{pp}}J_N^{(l-L)}\right) + \mathcal{E}\end{aligned}$$

Taking an expectation and rewriting, we get

$$\begin{aligned}\mathbb{E}\{T_3\} = &-\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\eta\left(\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_p^\perp \mathbf{Q_{fp}}J_N^{(l)}\right)\Big\} \\ &-\sum_{l=-(L-1)}^{L-1}\mathbb{E}\Big\{\eta\left(\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_f^\perp \mathbf{Q_{pp}}J_N^{(l-L)}\right)\Big\} \\ &+\mathbb{E}\{\mathcal{E}\} + T_3^{\mathcal{E}},\end{aligned}$$

where, as above, $T_3^{\mathcal{E}}$ is the term corresponding to $\left(\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}^\circ$ and $\left(\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\right)_{rs}^\circ$. According to Lemma 5.4, $\mathbb{E}\{\eta\Pi_p^\perp \mathbf{Q_{fp}}\}$ and $\mathbb{E}\{\eta\Pi_f^\perp \mathbf{Q_{pp}}\}$ are diagonal, it means that traces of these matrices multiplied by $J_N^{(k)}$ for $k \ne 0$ are zeros. Then

$$\mathbb{E}\{T_3\} = -\mathbb{E}\Big\{\eta\left(\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\right)_{rs}\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_p^\perp \mathbf{Q_{fp}}\right)\Big\} + \mathbb{E}\{\mathcal{E}\} + T_3^{\mathcal{E}}. \quad (5.37)$$

Finally, the term T_4 again consist factor $\frac{\partial \eta}{\partial W_{i_3u}^{m_3}}$ so as before, since $\mathbb{E}\{|{\mathbf{Q_{pp}}}^{rt}(W_pW_p^*)^{-1}|\mathbf{1}_\Omega\}$ is bounded, we can conclude that $\mathbb{E}\{T_4\}$ is of order $\mathcal{O}_{z^2}(N^{-k})$ for each k, and thus can be considered as the term $\mathbb{E}\{\mathcal{E}_N\}$.

Now we define $N \times N$ matrix $\mathbf{\Delta_{pp}}$ containing the error terms, i.e. with elements $\mathbf{\Delta}_{rs}(pp) = \mathbb{E}\{T_1^{\mathcal{E}} + T_2^{\mathcal{E}} + T_3^{\mathcal{E}} + \mathbb{E}\{\mathcal{E}\} + \mathcal{O}(\frac{1}{N^2})\}$, without taking into consideration factor $(1 - c_N)$, and by combining (5.30), (5.32), (5.37) we get that $\mathbb{E}\{\mathbf{Q_{pp}}\eta\Pi_p\}$ becomes

$$\mathbb{E}\Big\{\mathbf{Q_{pp}}\eta\Pi_p\Big\} = \frac{c_N}{1-c_N}\mathbb{E}\Big\{\mathbf{Q_{pp}}\Big\} - \frac{c_N}{1-c_N}\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}\Pi_p\Big\}$$
$$- \mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_p^{\perp}\mathbf{Q_{fp}}\right)\Big\} + \mathbf{\Delta_{pp}}$$

From what immediately follows

$$\mathbb{E}\Big\{\mathbf{Q_{pp}}\eta\Pi_p\Big\} = c_N\mathbb{E}\Big\{\mathbf{Q_{pp}}\Big\} - (1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_p^{\perp}\mathbf{Q_{fp}}\right)\Big\} + \mathbf{\Delta_{pp}} \tag{5.38}$$

Repeating step by step the above calculations we can get the analogous formula for $\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_p\}$:

$$\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_p\} = c_N\mathbb{E}\Big\{\mathbf{Q_{pf}}\Big\} - (1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_p^{\perp}\mathbf{Q_{ff}}\right)\Big\} + \mathbf{\Delta_{pf}} \tag{5.39}$$

Lemma 5.6. *The matrices $\mathbf{\Delta_{pp}}$ and $\mathbf{\Delta_{pf}}$ are diagonal and for $i = 1, \ldots, N$*

$$\mathbf{\Delta}_{ii}(pp) = \mathcal{O}_{z^2}\left(\frac{1}{N^{3/2}}\right) \tag{5.40}$$

$$\mathbf{\Delta}_{ii}(pf) = \mathcal{O}_{z^2}\left(\frac{1}{N^{3/2}}\right) \tag{5.41}$$

moreover, $N^{-1}\mathrm{Tr}\mathbf{\Delta_{pp}} = \mathcal{O}_{z^2}(N^{-2})$ and $N^{-1}\mathrm{Tr}\mathbf{\Delta_{pf}} = \mathcal{O}_{z^2}(N^{-2})$.

Proof. Due to the Lemma 3.6 all terms of equations (5.38) and (5.39) except of $\mathbf{\Delta_{pp}}$ and $\mathbf{\Delta_{pf}}$ are diagonal, what brings that $\mathbf{\Delta_{pp}}$ and $\mathbf{\Delta_{pf}}$ are also diagonal. The evaluation part is postponed to the Section 5.1.4.

On the other side, we recall that $\mathbf{Q_{pf}} = Q(z^2)\eta\Pi_p$ (see (5.9)), from what follows that $\mathbf{Q_{pf}}\eta\Pi_p = \eta\mathbf{Q_{pf}}$. Then equation (5.39) becomes

$$\mathbb{E}\{\mathbf{Q_{pf}}\} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right) = c_N\mathbb{E}\Big\{\mathbf{Q_{pf}}\Big\} - (1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\mathbf{Q_{ff}}\Pi_p^{\perp}\right)\Big\} + \mathbf{\Delta_{pf}}. \tag{5.42}$$

Now we express $\mathbf{Q_{ff}}\eta\Pi_p^{\perp}$ using (5.9) and resolvent identity

$$\mathbf{Q_{ff}}\eta\Pi_p^{\perp} = z(\eta^2\Pi_f\Pi_p - z^2)^{-1}\eta\Pi_p^{\perp} = z\left(-\frac{1}{z^2}\eta\Pi_p^{\perp} + \frac{1}{z^2}(\eta^2\Pi_f\Pi_p - z^2)^{-1}\eta^3\Pi_f\Pi_p\Pi_p^{\perp}\right) = -\frac{1}{z}\eta\Pi_p^{\perp} \tag{5.43}$$

We remind that $N^{-1}\mathrm{Tr}\Pi_p = c_N$ from what we get immediately $\mathbb{E}\{N^{-1}\mathrm{Tr}\mathbf{Q_{ff}}\eta\Pi_p^{\perp}\} = -\frac{(1-c_N)}{z} + \mathcal{O}(\frac{1}{N^k})$ for each k. This allows us to obtain expression for $\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}$ from (5.42) :

$$\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\} = \frac{z}{1-c_N}\mathbb{E}\{\mathbf{Q_{pf}}\} + \frac{z}{(1-c_N)^2}\mathbf{\Delta_{pf}} + \mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\}\mathcal{O}_{z^2}\left(\frac{1}{N^k}\right).$$

Let us rewrite $z\mathbf{Q_{pf}}$ as $\eta\mathbf{Q_{pp}}\Pi_p$ (see (5.9)) and notice that $\|\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\| = \mathcal{O}_{z^2}(1)$ (see Lemma 5.3 and the fact that $\|\eta W_p\|, \|\eta(W_pW_p^*)^{-1}\| \leq \kappa$). Thus, each element of $\mathbb{E}\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\}\mathcal{O}_{z^2}(\frac{1}{N^k})$ is $\mathcal{O}_{z^2}(\frac{1}{N^k})$ and the last expression becomes

$$\mathbb{E}\Big\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\Big\} = \frac{1}{1-c_N}\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\} + \frac{z}{(1-c_N)^2}\mathbf{\Delta_{pf}} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^k}\right). \tag{5.44}$$

By putting this into (5.38), after some easy calculations we get :

$$\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\}\left(1+N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pp}}\} - N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\} \times\left(\frac{z}{1-c_N}\mathbf{\Delta_{pf}} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^k}\right)\right) + \mathbf{\Delta_{pp}} \quad (5.45)$$

Since $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\} = N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}}\Pi_p^{\perp})\}$ and as was discussed in Corollary 5.1 $N^{-1}\mathbb{E}\{\mathrm{Tr}(\mathbf{Q_{fp}})\}$ is a Stieltjes transform at z^2 of some positive measure, we have that $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}$ is also a Stieltjes transform at z^2 of some positive measure. In particular it means that $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\} \leq \kappa(\rho(z^2))^{-1}$ and as consequence $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\} = \mathcal{O}_{z^2}(1)$. Thus (5.45) becomes

$$\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\}\left(1+N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}(1)(\mathbf{\Delta_{pf}}+\mathbf{\Delta_{pp}}) + \mathcal{O}_{z^2}^N\left(\frac{1}{N^k}\right)$$

We take normalized trace from both sides of obtained equation and notice that due to the Lemma 5.6 both, $N^{-1}\mathrm{Tr}\mathbf{\Delta_{pp}}$ and $N^{-1}\mathrm{Tr}\mathbf{\Delta_{pf}}$, are of order $\mathcal{O}_{z^2}(N^{-2})$

$$\frac{1}{N}\mathbb{E}\{\mathrm{Tr}\eta\mathbf{Q_{pp}}\Pi_p\}\left(1+N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}\right) = \frac{c_N}{N}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^2}\right) \quad (5.46)$$

Finally, to complete this paragraph, we denote

$$\tilde{\boldsymbol{\alpha}}_N = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pp}}\} = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{ff}}\} \quad (5.47)$$

$$\boldsymbol{\alpha}_N = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{pf}}\} = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{fp}}\} \quad (5.48)$$

and express $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}$, $N^{-1}\mathbb{E}\{\mathrm{Tr}\eta\mathbf{Q_{pp}}\Pi_p\}$ in terms of $\tilde{\boldsymbol{\alpha}}$. For this we use again the fact that $\mathbf{Q_{fp}} = \eta\Pi_f(\eta^2\Pi_p\Pi_f - z^2)^{-1}$ and write

$$N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\} = N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\mathbf{Q_{fp}})\} - N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta^2\Pi_p\Pi_f(\eta^2\Pi_p\Pi_f - z^2)^{-1})\}$$
$$= \boldsymbol{\alpha} - 1 - zN^{-1}\mathbb{E}\{\mathrm{Tr}(\mathbf{Q_{pp}})\} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right) = \boldsymbol{\alpha} - 1 - z\tilde{\boldsymbol{\alpha}} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right) \quad (5.49)$$

To deal with $N^{-1}\mathbb{E}\{\mathrm{Tr}\eta\mathbf{Q_{pp}}\Pi_p\}$ we simply remind that $\eta\mathbf{Q_{pp}}\Pi_p = z\mathbf{Q_{pf}}$. Now what is left is to find the connection between $\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\alpha}}$. We have

$$\frac{1}{N}\mathrm{Tr}\eta\Pi_f^{\perp}\mathbf{Q_{pp}} = \frac{1}{N}\mathrm{Tr}\left(\eta\mathbf{Q_{pp}} - \eta_N\Pi_f z(\eta^2\Pi_p\Pi_f - z^2)^{-1}\right) = \frac{1}{N}\mathrm{Tr}\left(\eta\mathbf{Q_{pp}} - z\mathbf{Q_{fp}}\right)$$

Taking the expectation from both sides and replacing in the first term η by 1 we get

$$\frac{1}{N}\mathbb{E}\{\eta\mathrm{Tr}\Pi_f^{\perp}\mathbf{Q_{pp}}\} = \tilde{\boldsymbol{\alpha}} - z\boldsymbol{\alpha} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right) \quad (5.50)$$

for each k.
On the other hand, using (5.9), resolvent identity and the fact that $\Pi_f\Pi_f^{\perp} = 0$ we get

$$\frac{1}{N}\mathrm{Tr}\eta_N\Pi_f^{\perp}\mathbf{Q_{pp}} = \frac{1}{N}\mathrm{Tr}\left(z\left(-\frac{1}{z^2} + \frac{1}{z^2}(\eta_N^2\Pi_p\Pi_f - z^2)^{-1}\eta_N^2\Pi_p\Pi_f\right)\eta_N\Pi_f^{\perp}\right) = -\frac{1}{zN}\mathrm{Tr}\eta_N\Pi_f^{\perp}$$

since $N^{-1}\mathrm{Tr}\Pi_f = c_N$, we conclude immediately

$$\frac{1}{N}\mathbb{E}\{\eta_N\mathrm{Tr}\Pi_f^{\perp}\mathbf{Q_{pp}}\} = -\frac{1-c_N}{z} + \mathcal{O}\left(\frac{1}{N^k}\right) \quad (5.51)$$

Compare the last expression to (5.50) we finally obtain after trivial algebra

$$\boldsymbol{\alpha}_N(z) = \frac{\tilde{\boldsymbol{\alpha}}_N(z)}{z} + \frac{1-c_N}{z^2} + \mathcal{O}_{z^2}\left(\frac{1}{N^k}\right) \quad (5.52)$$

Since, in (5.46) $\eta\mathbf{Q_{pp}}\Pi_p = z\mathbf{Q_{pf}}$, by combining it with (5.52) and (5.49) we obtain the equation for $\tilde{\boldsymbol{\alpha}}$:

$$(1-z^2)\tilde{\boldsymbol{\alpha}}_N^2 + \left(\frac{2(1-c_N)}{z} - z\right)\tilde{\boldsymbol{\alpha}}_N + \frac{(1-c_N)^2}{z^2} = \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right). \tag{5.53}$$

Also if we denote $\tilde{\alpha}_N(z) = N^{-1}\mathbb{E}\{\mathrm{Tr}Q_N(z)\}$ and $\alpha_N(z) = N^{-1}\mathbb{E}\{\mathrm{Tr}\eta\Pi_p Q_N(z)\}$, then $\tilde{\boldsymbol{\alpha}}_N(z) = z\tilde{\alpha}_N(z^2)$ and $\boldsymbol{\alpha}_N(z) = \alpha_N(z^2)$. The respective equation for $\tilde{\alpha}_N(z)$ will be

$$(1-z^2)z^2\tilde{\alpha}_N^2(z^2) + \left(2(1-c_N) - z^2\right)\tilde{\alpha}_N(z^2) + \frac{(1-c_N)^2}{z^2} = \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right).$$

The l.h.s of obtained equation is the function of z^2, thus the error term of r.h.s is also necessary a function of z^2. By exchanging z^2 with z we have

$$(1-z)z\tilde{\alpha}_N^2(z) + (2(1-c_N) - z)\,\tilde{\alpha}_N(z) + \frac{(1-c_N)^2}{z} = \mathcal{O}_z\left(\frac{1}{N^2}\right). \tag{5.54}$$

Moreover, from (5.52) in the similar way one can easily deduce the corresponding expression on $\tilde{\alpha}_N(z)$ and $\alpha_N(z)$:

$$\alpha_N(z) = \tilde{\alpha}_N(z) + \frac{1-c_N}{z} + \mathcal{O}_z\left(\frac{1}{N^k}\right). \tag{5.55}$$

Remark 5.2. *It is easy to see that due to Corollary 5.1 α_N is a Stieltjes transform of a positive measure carried by $\mathbb{R}^+$ with mass $c_N + \mathcal{O}_z(N^{-k})$. Since $-\frac{1-c_N}{z}$ is a Stieltjes transform of measure $(1-c_N)\delta_0$, we conclude that $\tilde{\alpha}_N(z) = \alpha_N(z) - \frac{1-c_N}{z} + \mathcal{O}_z\left(N^{-k}\right)$ is a Stieltjes transform of a positive probability measure carried by $\mathbb{R}^+$ up to a term $\mathcal{O}_z(N^{-k})$ for each $k \in \mathbb{N}$.*

Finally, we prove here a useful Proposition.

Proposition 5.1. *Matrices $\mathbb{E}\{\mathbf{Q_{ij}}\}$, $\mathbb{E}\{\mathbf{Q_{ij}}\eta\Pi_k\}$ for $i,j,k \in \{p,f\}$ are multiple of I_N up to an error term.*

Proof. For the beginning we deal with $\mathbb{E}\{\mathbf{Q_{pp}}\}$. Before we obtain an equation (5.45) which along with Lemma 5.6 gives :

$$\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\}\left(1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right)$$

Since $\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\} = z\mathbb{E}\{\mathbf{Q_{pf}}\}$, last equation becomes

$$z\mathbb{E}\{\eta\mathbf{Q_{pf}}\}\left(1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^{\perp}\mathbf{Q_{fp}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \tag{5.56}$$

In order to find the structure of $\mathbb{E}\{\mathbf{Q_{pp}}\}$ we need to obtain one more equation that connects $\mathbb{E}\{\eta\mathbf{Q_{pp}}\}$ and $\mathbb{E}\{\eta\mathbf{Q_{pf}}\}$. For this we repeat steps that led to (5.56). Following the calculations with integration by parts formula for $\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\}$ and $\mathbb{E}\{\mathbf{Q_{pp}}\eta\Pi_f\}$ we obtain :

$$\mathbb{E}\Big\{\mathbf{Q_{pf}}\eta\Pi_f\Big\} = c_N\mathbb{E}\Big\{\mathbf{Q_{pf}}\Big\} - (1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_f^{\perp}\mathbf{Q_{pf}}\right)\Big\} + \boldsymbol{\Delta}^{\mathbf{1}}_{\mathbf{pf}} \tag{5.57}$$

$$\mathbb{E}\Big\{\mathbf{Q_{pp}}\eta\Pi_f\Big\} = c_N\mathbb{E}\Big\{\mathbf{Q_{pp}}\Big\} - (1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\Big\}\frac{1}{N}\mathbb{E}\Big\{\mathrm{Tr}\left(\eta\Pi_f^{\perp}\mathbf{Q_{pp}}\right)\Big\} + \boldsymbol{\Delta}^{\mathbf{1}}_{\mathbf{pp}}, \tag{5.58}$$

where error terms $\boldsymbol{\Delta}^{\mathbf{1}}_{\mathbf{pf}}$ and $\boldsymbol{\Delta}^{\mathbf{1}}_{\mathbf{pp}}$ are also satisfy Lemma 5.6. In the case with (5.38)-(5.39) we were able to obtain from (5.39) that $\mathbb{E}\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\} = \frac{1}{1-c_N}\mathbb{E}\{\eta\mathbf{Q_{pp}}\Pi_p\} + \mathcal{O}_{z^2}^N(N^{-3/2})$. So now we show that from (5.58) it can be deduced analogues expression, i.e. $(1-c_N)\mathbb{E}\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\} = \mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} + \mathcal{O}_{z^2}^N(N^{-3/2})$. Indeed, first let us remind that $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_f^{\perp}\mathbf{Q_{pp}})\} = -\frac{1-c_N}{z} + \mathcal{O}_{z^2}(N^{-k})$ for each k (see (5.51)). Since $\mathbb{E}\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\} = \mathcal{O}_{z^2}(1)$ we rewrite (5.58) as

$$(1-c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\Big\} = \frac{z}{1-c_N}\left(\mathbb{E}\Big\{\mathbf{Q_{pp}}\eta\Pi_f\Big\} - c_N\mathbb{E}\Big\{\mathbf{Q_{pp}}\Big\}\right) + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \tag{5.59}$$

Using resolvent identity and the facts that $\Pi_f^2 = \Pi_f$, $\mathbb{E}\{\eta_N \Pi_f\} = c_N I_N + \mathcal{O}_{z^2}(N^{-3/2})$ due to (5.20), we write

$$\mathbb{E}\{\mathbf{Q_{pp}}\eta\Pi_f\} = z\mathbb{E}\{(-\frac{1}{z^2} + \frac{1}{z^2}(\eta_N^2\Pi_p\Pi_f - z^2)^{-1}\eta_N^2\Pi_p\Pi_f)\eta_N\Pi_f\} = -\frac{1}{z}\mathbb{E}\{\eta_N\Pi_f\}$$
$$+ \frac{1}{z}\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} + \mathcal{O}_{z^2}(N^{-k}) = -\frac{c_N}{z}I_N + \frac{1}{z}\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} + \mathcal{O}_{z^2}(N^{-k})$$

for each k. Putting this into r.h.s. of (5.59) we have :

$$(1 - c_N)\mathbb{E}\Big\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\Big\} \;=\; \frac{1}{1-c_N}\left(-c_N\left(I_N + z\mathbb{E}\{\mathbf{Q_{pp}}\}\right) + \mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\}\right) + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right)$$

It is left to notice that $\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} = \mathbb{E}\{(\eta^2\Pi_p\Pi_f - z^2)^{-1}\eta^2\Pi_p\Pi_f\} = I_N + z\mathbb{E}\{\mathbf{Q_{pp}}\}$. Thus we immediately obtain that $(1 - c_N)\mathbb{E}\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f\} = \mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} + \mathcal{O}_{z^2}^N(N^{-3/2})$.
Now we put obtained expression of $(1-c_N)\mathbb{E}\{\eta\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-2}W_f$ into (5.57) and since $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_f^\perp\mathbf{Q_{pf}})\}$ coincides with $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\}$ (due to the symmetry, see Lemma 5.4) and $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\} = \mathcal{O}_{z^2}(1)$ we obtain

$$\mathbb{E}\{\eta\mathbf{Q_{pf}}\Pi_f\}\left(1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_f^\perp\mathbf{Q_{pf}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pf}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right)$$

or, taking into account $\mathbb{E}\{\mathbf{Q_{pf}}\eta\Pi_f\} = I_N + z\mathbb{E}\{\mathbf{Q_{pp}}\}$,

$$(I_N + z\mathbb{E}\{\mathbf{Q_{pp}}\})\left(1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_f^\perp\mathbf{Q_{pf}})\}\right) = c_N\mathbb{E}\{\mathbf{Q_{pf}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right)$$

Further for more convenience we denote the scalars $1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_f^\perp\mathbf{Q_{pf}})\} = 1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\}$ by $w_N(z)$ or simply w_N. Thus the last equation can be rewritten as

$$c_N\mathbb{E}\{\mathbf{Q_{pf}}\} = w_N + zw_N\mathbb{E}\{\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \tag{5.60}$$

Finally we have a system of equation (5.60), (5.56) for $\mathbb{E}\{\mathbf{Q_{pf}}\}$ and $\mathbb{E}\{\mathbf{Q_{pp}}\}$:

$$\begin{cases} c_N\mathbb{E}\{\mathbf{Q_{pf}}\} = w_N + zw_N\mathbb{E}\{\mathbf{Q_{pp}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \\ c_N\mathbb{E}\{\mathbf{Q_{pp}}\} = zw_N\mathbb{E}\{\mathbf{Q_{pf}}\} + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \end{cases} \tag{5.61}$$

By putting the first equation of (5.61) into second one multiplied by c_N we obtain with little algebra :

$$\mathbb{E}\{\mathbf{Q_{pp}}\}\left(c_N^2 - z^2w_N^2\right) = zw_N^2 I_N + \mathcal{O}_{z^2}^N\left(\frac{1}{N^{3/2}}\right) \tag{5.62}$$

To conclude from this equation the statement of the lemma we need to prove that $\left(c_N^2 - z^2w_N^2\right)^{-1} = \mathcal{O}_{z^2}(1)$. For this we write

$$\frac{1}{c_N^2 - z^2w_N^2} = -\frac{1}{z^2w_N\left(-\frac{c_N^2}{z^2w_N} + w_N\right)} \tag{5.63}$$

It is known that if $f(z) \in \mathcal{S}(\mathbb{R}^+)$, then $-\frac{1}{z(1+f(z))} \in \mathcal{S}(\mathbb{R}^+)$. Since $N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\} = N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\eta\Pi_p^\perp\}$ is a Stieltjes transform evaluated at z^2 of some positive measure curried by $\mathbb{R}^+$ (due to reasons similar to those in Corollary 5.1), this implies that $-(z^2w_N)^{-1} = -(z^2(1 + N^{-1}\mathbb{E}\{\mathrm{Tr}(\eta\Pi_p^\perp\mathbf{Q_{fp}})\}))^{-1}$ is also a Stieltjes transform evaluated at z^2 of some measure curried by $\mathbb{R}^+$, and as consequence we an say the same about $-z^{-2}(-\frac{c_N^2}{z^2w_N} + w_N)^{-1}$. So the absolute value of r.h.s. of (5.63) can be bounded with $|z|^2\rho^{-2}(z^2)$:

$$\left|\frac{1}{c_N^2 - z^2w_N^2}\right| = \left|\frac{1}{z^2w_N\left(-\frac{c_N^2}{z^2w_N} + w_N\right)}\right| = \left|\frac{1}{z^2w_N}\right||z|^2\left|\frac{1}{z^2(-\frac{c_N^2}{z^2w_N} + w_N)}\right| \leq \frac{\kappa|z|^2}{\rho^2(z^2)}$$

This means that we can divide by $c_N - \frac{z^2 w_N^2}{c_N}$ and obtain

$$\mathbb{E}\{\mathbf{Q_{pp}}\} = \frac{z w_N^2}{c_N^2 - z^2 w_N^2} I_N + \mathcal{O}_{z^2}^N \left(\frac{1}{N^{3/2}} \right)$$

Also this and the first equation of system (5.61) provide us with the expression for $\mathbb{E}\{\mathbf{Q_{pf}}\}$:

$$\mathbb{E}\{\mathbf{Q_{pf}}\} = \frac{c_N^2}{c_N^2 - z^2 w_N^2} I_N + \mathcal{O}_{z^2}^N \left(\frac{1}{N^{3/2}} \right)$$

5.1.3 Stieltjes transform and limiting distribution

Let us introduce the measure $\tilde{\nu}_N = (c_N \delta_1 + (1-c_N)\delta_0) \boxtimes (c_N \delta_1 + (1-c_N)\delta_0)$, where δ_x is the Dirac measure at the point x, and denote by $\tilde{t}_N$ its Stieltjes transform. The goal of this Section is to prove that $\tilde{\alpha}_N - \tilde{t}_N \to 0$ for $N \to +\infty$.
The form of $\tilde{t}_N$ and $\tilde{\nu}_N$ is known (see for example Example 3.6.7. [49]). In particular it appears that $\tilde{t}_N$ satisfies the equation (5.54), but in which the term $\mathcal{O}_z(N^{-2})$ is replaced 0, i.e.

$$z(1-z)\tilde{t}_N^2(z) + (2(1-c_N) - z)\tilde{t}_N(z) + \frac{(1-c_N)^2}{z} = 0 \tag{5.64}$$

In order to evaluate $\tilde{\alpha}_N - \tilde{t}_N$ it is natural to take a difference between equations (5.54) and (5.64) :

$$(\tilde{\alpha}_N - \tilde{t}_N)((1-z)z(\tilde{\alpha}_N + \tilde{t}_N) + 2(1-c_N) - z) = \mathcal{O}_z \left(\frac{1}{N^2} \right)$$

We remind that $\tilde{\alpha}_N = \alpha_N - \frac{1-c_N}{z} + \mathcal{O}_z(N^{-k})$ (see (5.55)) and rewrite the l.h.s. of the last equation :

$$(\tilde{\alpha}_N - \tilde{t}_N)((1-z)z\alpha_N - (1-z)(1-c_N) + (1-z)z\tilde{t}_N + 2(1-c_N) - z + \mathcal{O}_z(N^{-k})) = \mathcal{O}_z \left(\frac{1}{N^2} \right)$$

$$\Downarrow$$

$$(\tilde{\alpha}_N - \tilde{t}_N)((1-z)z\alpha_N - (1-z)(1-c_N) + (1-z)z\tilde{t}_N + 2(1-c_N) - z) + (\tilde{\alpha}_N - \tilde{t}_N)\mathcal{O}_z \left(\frac{1}{N^k} \right) = \mathcal{O}_z \left(\frac{1}{N^2} \right)$$

Since α_N is a Stieltjes transform then $|\alpha_N| = \mathcal{O}_z(1)$ and from (5.55) we have $|\tilde{\alpha}_N| = \mathcal{O}_z(1)$, also $\tilde{t}_N$ is a Stieltjes transform thus bounded by $\rho^{-1}(z)$. This means that $(\tilde{\alpha}_N - \tilde{t}_N)\mathcal{O}_z(N^{-k}) = \mathcal{O}_z(N^{-k})$. Finally we obtain the expression for $(\tilde{\alpha}_N - \tilde{t}_N)$:

$$\tilde{\alpha}_N - \tilde{t}_N = \frac{\mathcal{O}_z\left(N^{-2}\right)}{(1-z)z\alpha_N - (1-z)(1-c_N) + (1-z)z\tilde{t}_N + 2(1-c_N) - z}$$

To evaluate denominator we return to (5.64) and write :

$$(1-z)z\tilde{t}_N + 2(1-c_N) - z = -\frac{(1-c_N)^2}{z\tilde{t}_N}$$

Moreover, since $\tilde{t}_N$ is the Stieltjes transform of a positive measure curried by $\mathbb{R}^+$, $\text{Im} z\tilde{t}_N > 0$ for $z \in \mathbb{C}^+$ and we also get that $\text{Im}((1-z)z\tilde{t}_N) = \text{Im} z - \text{Im}\frac{(1-c_N)^2}{z\tilde{t}_N} > \text{Im} z$. Now we rewrite denominator as

$$(1-z)z\alpha_N - (1-z)(1-c_N) + (1-z)z\tilde{t}_N + 2(1-c_N) - z = (1-z)\left(z\alpha_N - (1-c_N) - \frac{(1-c_N)^2}{(1-z)z\tilde{t}_N} \right)$$

and notice that due to discussion in Corollary 5.1 α_N also is the Stieltjes transform of a positive measure curried by $\mathbb{R}^+$, so $\text{Im} z\alpha_N > 0$ for $z \in \mathbb{C}^+$. Thus

$$|(1-z)z\alpha_N - (1-z)(1-c_N) + (1-z)z\tilde{t}_N + 2(1-c_N) - z| \geq |1-z|\text{Im}\frac{-(1-c_N)^2}{(1-z)z\tilde{t}_N} = \frac{(1-c_N)^2 \text{Im}((1-z)z\tilde{t}_N)}{|1-z||z|^2|\tilde{t}_N|^2}$$

Finally, we remind that $\mathrm{Im}((1-z)z\tilde{t}_N) > \mathrm{Im}z$ and $\tilde{t}_N \leq (\mathrm{Im}z)^{-1}$ and conclude that on $\mathbb{C}^+$

$$\tilde{\alpha}_N(z) - \tilde{t}_N(z) \leq \frac{\mathcal{O}_z\left(N^{-2}\right)(1+|z|)|z|^2}{(1-c_N)^2(\mathrm{Im}z)^3} = \mathcal{O}_z\left(N^{-2}\right). \tag{5.65}$$

We introduce here the expression for $\tilde{t}_N$ on $z \in \mathbb{C}^+$

$$\tilde{t}_N(z) = \frac{z - 2(1-c_N) + \sqrt{z(z - 4c_N(1-c_N))}}{2(1-z)z}$$

where we define function $z \mapsto \sqrt{z}$ for $z = |z|e^{i\theta}$, $\theta \in (0, 2\pi)$ as $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$. In particular, if $x \in \mathbb{R}$ and $z = xe^{i\theta}$ then $\sqrt{z} \xrightarrow[\theta \searrow 0]{} \sqrt{x}$ and $\sqrt{z} \xrightarrow[\theta \nearrow 2\pi]{} -\sqrt{x}$. Then one can easily obtain that there exists $\lim_{z \to x, z \in \mathbb{C}^+} \tilde{t}_N(z)$ for $x \in (-\infty, 0) \cap (4c_N(1-c_N), +\infty)$ and $x \neq 1$, that we will denote by $\tilde{t}_N(x)$ and

$$\tilde{t}_N(x) = \begin{cases} \dfrac{x - 2(1-c_N) - \sqrt{x(x - 4c_N(1-c_N))}}{2(1-x)x}, & x < 0 \\ \dfrac{x - 2(1-c_N) + \sqrt{x(x - 4c_N(1-c_N))}}{2(1-x)x}, & x > 4c_N(1-c_N), x \neq 1 \end{cases} \tag{5.66}$$

Moreover, $\tilde{t}_N(x)$ is a solution of equation (5.64) with z replaced by x. It is also known that $\tilde{\nu}_N = (c_N\delta_1 + (1-c_N)\delta_0) \boxtimes (c_N\delta_1 + (1-c_N)\delta_0)$ is defined as

$$d\tilde{\nu}_N(\lambda) = \frac{\sqrt{\lambda(4c_N(1-c_N) - \lambda)}}{2\pi\lambda(1-\lambda)} \mathbf{1}_{[0, 4c_N(1-c_N)]} d\lambda + (1-c_N)\delta_\lambda + \max(2c_N - 1, 0)\delta_{\lambda - 1} \tag{5.67}$$

Besides, it is easy to see that measure $\nu_N = \frac{1}{c_N}\tilde{\nu}_N - \frac{1-c_N}{c_N}\delta_0$ is also a positive probability measure carried by $\mathbb{R}^+$ with corresponding Stieltjes transform $t_N(z) = \frac{\tilde{t}_N(z)}{c_N} + \frac{1-c_N}{c_N z}$ and

$$t_N(z) = \frac{z(2c_N - 1) + \sqrt{z(z - 4c_N(1-c_N))}}{2c_N(1-z)z}, \quad z \in \mathbb{C}^+$$

$$t_N(x) = \begin{cases} \dfrac{x(2c_N - 1) - \sqrt{x(x - 4c_N(1-c_N))}}{2c_N(1-x)x}, & x < 0 \\ \dfrac{x(2c_N - 1) + \sqrt{x(x - 4c_N(1-c_N))}}{2c_N(1-x)x}, & x > 4c_N(1-c_N), x \neq 1 \end{cases} \tag{5.68}$$

We deduce several immediate but useful facts.

Remark 5.3. *Since $\alpha_N(z) = \tilde{\alpha}_N(z) + \frac{1-c_N}{z} + \mathcal{O}_z(N^{-k})$ for each k, we can conclude that*

$$\alpha_N(z) - c_N t_N(z) \to 0$$

for $z \in \mathbb{C}^+$. Also if we denote $\mathbf{t}_N(z) = t_N(z^2)$, $\tilde{\mathbf{t}}_N(z) = z\tilde{t}_N(z^2)$, we have immediately for $z^2 \in \mathbb{C}^+$

$$\boldsymbol{\alpha}_N(z) - c_N \mathbf{t}_N(z) \to 0 \, a.s. \tag{5.69}$$

$$\tilde{\boldsymbol{\alpha}}_N(z) - \tilde{\mathbf{t}}_N(z) \to 0 \, a.s. \tag{5.70}$$

Moreover, due to the Proposition 3.4 $\tilde{\mathbf{t}}_N \in \mathcal{S}(\mathbb{R})$ and

$$\mathbf{t}(z)_N = \frac{\tilde{\mathbf{t}}_N(z)}{c_N z} + \frac{1-c_N}{c_N z^2} \tag{5.71}$$

Corollary 5.2. *The empirical eigenvalue distribution $\hat{\nu}_N$ of $\Pi_{p,N}\Pi_{f,N}$ verifies*

$$\hat{\nu}_N - \tilde{\nu}_N \to 0 \tag{5.72}$$

weakly almost surely.

Proof. Above we proved that $\mathbb{E}\{\frac{1}{N}\text{Tr}Q_N(z)\} - \tilde{t}_N(z) \to 0$ for each $z \in \mathbb{C}^+$. The Poincaré-Nash inequality and the Borel Cantelli Lemma imply that $\frac{1}{N}\text{Tr}(Q_N(z)) - \mathbb{E}\{\frac{1}{N}\text{Tr}Q_N(z)\} \to 0$ a.s. for each $z \in \mathbb{C} - \mathbb{R}^+$. Therefore, it holds that

$$\frac{1}{N}\text{Tr}(Q_N(z)) - \tilde{t}_N(z) \to 0 a.s. \tag{5.73}$$

for each $z \in \mathbb{C}^+$. Corollary 2.7 of [18] implies that $\hat{\nu}_N - \tilde{\nu}_N \to 0$ weakly almost surely provided we verify that $(\hat{\nu}_N)_{N\geq 1}$ is almost surely tight and that $(\tilde{\nu}_N)_{N\geq 1}$ is tight. Since $\tilde{\nu}_N$ is a multiplicative free convolution of Dirac measures, it is known that $(\tilde{\nu}_N)_{N\geq 1}$ is tight. For $(\hat{\nu}_N)_{N\geq 1}$ we write

$$\int_{\mathbb{R}^+} \lambda \, d\,\hat{\nu}_N(\lambda) = \frac{1}{N}\text{Tr}\Pi_{p,N}\Pi_{f,N} \leq 1$$

almost surely. This implies that $(\hat{\nu}_N)_{N\geq 1}$ is almost surely tight. ■

Also since $\tilde{\nu}_N$ is a deterministic equivalent of the empirical eigenvalue distribution of $\Pi_{p,N}\Pi_{f,N}$, it immediately follows that ν_N is a deterministic equivalent of the empirical eigenvalue distribution of $(\hat{R}^L_{f,y})^{-1/2}\hat{R}^L_{f|p,y}(\hat{R}^L_{p,y})^{-1}\hat{R}^{L*}_{f|p,y}(\hat{R}^L_{f,y})^{-1/2}$.

We define the support of $\tilde{\nu}_N$ by $\mathcal{S}_N$, obviously, it coincides with the support of ν_N and $\mathcal{S}_N = [0, 4c_N(1-c_N)] \cup \{1\}\mathbf{1}_{c_N>1/2}$. Moreover, the support of corresponding measure of $\tilde{\mathbf{t}}_N$ is $\boldsymbol{\mathcal{S}}_N = [-\sqrt{4c_N(1-c_N)}, \sqrt{4c_N(1-c_N)}] \cup \{\pm 1\}\mathbf{1}_{c_N>1/2}$. While $\mathbf{t}_N$ is not a Stieltjes transform, we can however say that $\mathbf{t}_N$ is also holomorphic outside $\boldsymbol{\mathcal{S}}_N$.

5.1.4 Proof of Lemma 5.6

To evaluate $\boldsymbol{\Delta}_{\mathbf{pp}}$ and $\boldsymbol{\Delta}_{\mathbf{pf}}$ we first should prove the next lemma which is based on the Poincaré-Nash inequality.

Lemma 5.7. *Let $(F_N)_{N\geq 1}$ and $(G_N)_{N\geq 1}$ be sequences of deterministic $N\times N$ matrices such that $\sup_N \|F_N\|$, $\sup_N \|G_N\| \leq \kappa$, and consider sequences of deterministic N–dimensional vectors $(a_{1,N})_{N\geq 1}$, $(a_{2,N})_{N\geq 1}$ such that $sup_N\|a_{i,N}\| \leq \kappa$ for $i = 1, 2$. Then, for each $z \in \mathbb{C}^+$ and $i, j, h = \{p, f\}$, it holds that*

$$\mathbf{Var}\left\{\frac{1}{N}\text{Tr}F\mathbf{Q_{ij}}\right\} = \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right), \tag{5.74}$$

$$\mathbf{Var}\left\{\frac{1}{N}\text{Tr}\mathbf{Q_{ij}}F\eta_N\Pi_h G\right\} = \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right), \tag{5.75}$$

$$\mathbf{Var}\left\{\frac{1}{N}\text{Tr}\mathbf{Q_{ij}}F\eta_N\Pi_h^{\perp} G\right\} = \mathcal{O}_{z^2}\left(\frac{1}{N^2}\right), \tag{5.76}$$

$$\mathbf{Var}\left(a_1^*\eta\mathbf{Q_{ij}}W_h^*(W_hW_h^*)^{-1}F(W_kW_k^*)^{-1}W_k a_2\right) = \mathcal{O}_{z^2}\left(\frac{1}{N}\right), \tag{5.77}$$

$$\mathbf{Var}\left\{a_1^*\mathbf{Q_{ij}}a_2\right\} = \mathcal{O}_{z^2}\left(\frac{1}{N}\right), \tag{5.78}$$

$$\mathbf{Var}\left\{a_1^*\mathbf{Q_{ij}}F\eta_N\Pi_h a_2\right\} = \mathcal{O}_{z^2}\left(\frac{1}{N}\right). \tag{5.79}$$

where $C(z)$ can be written as $C(z) = P_1(|z|)P_2\left(\frac{1}{\text{Im}z}\right)$ for some nice polynomials P_1 and P_2.

Proof. We first prove (5.74) for $\mathbf{Q_{pp}}$ and denote by ξ the term $\xi = \frac{1}{N}\text{Tr}F\mathbf{Q_{pp}}$. The Poincaré-Nash inequality (2.3) leads to

$$\mathbf{Var}\{\xi\} \leq \sum_{\substack{i_1,j_1,m_1\\ i_2,j_2,m_2}} \mathbb{E}\left\{\left(\frac{\partial\xi}{\partial\overline{W}^{m_1}_{i_1,j_1}}\right)^* \mathbb{E}\{W^{m_1}_{i_1,j_1}\overline{W}^{m_2}_{i_2,j_2}\}\frac{\partial\xi}{\partial\overline{W}^{m_2}_{i_2,j_2}}\right\}$$
$$+ \sum_{\substack{i_1,j_1,m_1\\ i_2,j_2,m_2}} \mathbb{E}\left\{\frac{\partial\xi}{\partial W^{m_1}_{i_1,j_1}}\mathbb{E}\{W^{m_1}_{i_1,j_1}\overline{W}^{m_2}_{i_2,j_2}\}\left(\frac{\partial\xi}{\partial W^{m_2}_{i_2,j_2}}\right)^*\right\}.$$

We just evaluate the second term of r.h.s., denoted by ϕ. Since F does not depend on W, derivative of ξ can be found with help of (5.35) :

$$\frac{\partial \xi}{\partial W_{i_1 j_1}^{m_1}} = -\frac{\eta}{N}\mathrm{Tr}F\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_1}^{m_1}\mathbf{e}_{j_1}^*\Pi_p^\perp\mathbf{Q_{fp}}\mathbf{1}_{i_1\le L} \\ -\frac{\eta}{N}\mathrm{Tr}F\mathbf{Q_{pf}}W_f^*(W_fW_f^*)^{-1}\mathbf{f}_{i_1-L}^{m_1}\mathbf{e}_{j_1}^*\Pi_f^\perp\mathbf{Q_{pp}}\mathbf{1}_{i_1>L}+\mathcal{O}\left(\frac{1}{N^k}\right)$$

It is easy to see that ϕ composed with four similar parts of the form

$$\frac{1}{N^3}\sum_{\substack{i_1,j_1,m_1\\ i_2,j_2,m_2}}\delta_{m_1,m_2}\delta_{i_1+j_1,i_2+j_2}\mathbb{E}\left\{\eta^2\mathbf{e}_{j_1}^*\Pi_p^\perp\mathbf{Q_{fp}}F\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}\mathbf{f}_{i_1}^{m_1}\mathbf{f}_{i_2}^{m_2*}(W_pW_p^*)^{-1}W_p\mathbf{Q}_{\mathbf{pp}}^*F\mathbf{Q}_{\mathbf{fp}}^*\Pi_p^\perp\mathbf{e}_{j_2}\right\} \tag{5.80}$$

where $1\le i_1,i_2\le L$. Now we again denote $l=i_1-i_2=j_2-j_1$ which lies in $(-L+1,L-1)$ and remark that $\sum_{m_1,m_2,i_1,i_2}\delta_{m_1,m_2}\delta_{i_1-i_2,l}\mathbf{f}_{i_1}^{m_1}\mathbf{f}_{i_2}^{m_2*}=(J_L^{(l)}\otimes I_M)$ as well as $\sum_{j_1,j_2}\delta_{j_2-j_1,l}\mathbf{e}_{j_2}\mathbf{e}_{j_1}^*=J_N^{(l)}$ this allow us to rewrite last expression as

$$\frac{1}{N^3}\sum_{l=-(L-1)}^{L-1}\mathbb{E}\left\{\eta^2\mathrm{Tr}\Pi_p^\perp\mathbf{Q_{fp}}F\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-1}(J_L^{(l)}\otimes I_M)(W_pW_p^*)^{-1}W_p\mathbf{Q}_{\mathbf{pp}}^*F^*\mathbf{Q}_{\mathbf{fp}}^*\Pi_p^\perp J_N^{(l)}\right\} \tag{5.81}$$

For each $N\times ML$ matrices A and B, the Schwartz inequality and the inequality between arithmetic and geometric means lead to

$$\left|\frac{1}{N}\mathrm{Tr}A(I_M\otimes J_L^{*(l)})B^*J_N^{*(l)}\right|\le\frac{1}{2N}\mathrm{Tr}A(I_M\otimes J_L^{*(l)}J_L^{(l)})A^*+\frac{1}{2N}\mathrm{Tr}BJ_N^{*(l)}J_N^{(l)}B^*.$$

Therefore, since $I_M\otimes J_L^{*(l)}J_L^{(l)}\le I_{ML}$ and $J_N^{*(l)}J_N^{(l)}\le I_N$

$$\left|\frac{1}{N}\mathrm{Tr}A(I_M\otimes J_L^{*(l)})B^*J_N^{*(l)}\right|\le\frac{C}{2N}(\mathrm{Tr}A^*A+\mathrm{Tr}B^*B). \tag{5.82}$$

We take $A=B=\Pi_p^\perp\mathbf{Q_{fp}}F\mathbf{Q_{pp}}W_p^*\eta(W_pW_p^*)^{-1}$, then what is left is to bound $N^{-1}\mathbb{E}\{\mathrm{Tr}AA^*\}$. Since $\mathbf{Q}_{\mathbf{pp}}^*F^*\mathbf{Q}_{\mathbf{fp}}^*\Pi_p^\perp\mathbf{Q_{fp}}F\mathbf{Q_{pp}}\le\kappa^2\|\mathbf{Q_{pp}}\|^2\|\mathbf{Q_{fp}}\|^2I_N$ and $\|\mathbf{Q_{pp}}\|,\|\mathbf{Q_{fp}}\|=\mathcal{O}_z^2(1)$ we have with $\eta^2(W_pW_p^*)^{-2}\le((1-\sqrt{c_*})^2-2\epsilon)^{-2}I_{ML}$ (see (5.4)) :

$$\frac{1}{N}\mathbb{E}\left\{\mathrm{Tr}\Pi_p^\perp\mathbf{Q_{fp}}F\mathbf{Q_{pp}}W_p^*\eta^2(W_pW_p^*)^{-2}W_p\mathbf{Q}_{\mathbf{pp}}^*F^*\mathbf{Q}_{\mathbf{fp}}^*\Pi_p^\perp\right\}\le\mathcal{O}_z^2(1)\mathbb{E}\left\{\|W_p\|^2\right\}=\mathcal{O}_z^2(1) \tag{5.83}$$

Taking into account that L is constant, this gives us immediately

$$\phi=\mathcal{O}_z^2\left(\frac{1}{N^2}\right)$$

which finishes the proof of (5.78). Obviously for $\mathbf{Q_{ff}},\mathbf{Q_{pf}}$, etc. the proof is analogous.
To proof (5.75)-(5.76) we follow the same scheme. If we take $\xi=\frac{1}{N}\mathrm{Tr}\mathbf{Q_{pp}}F\eta_N\Pi_pG$ then, after some calculation we come to the step where we need to evaluate $N^{-1}\mathbb{E}\{\mathrm{Tr}AA^*\}$ with $A=\Pi_p^\perp\mathbf{Q_{fp}}F\eta_N\Pi_pG\mathbf{Q_{pp}}W_p^*\eta(W_pW_p^*)^{-1}$ and $A=\eta_N\Pi_p^\perp G\mathbf{Q_{pp}}W_p^*\eta(W_pW_p^*)^{-1}$. As we can see, these expressions are similar to (5.83) and can be evaluated in the same way, thus we omit further explanation.
(5.78) is the consequence of (5.74) since $a_1^*\mathbf{Q}a_2=\mathrm{Tr}\mathbf{Q}a_2a_1^*=\mathrm{Tr}\mathbf{Q}F$ for $F=a_2a_1^*$. Analogous, (5.79) is the consequence of (5.75). This completes the proof of Lemma 5.7. ■
Now we can return to the proof of Lemma 5.6. We will focus on the $\mathbf{\Delta_{pp}}$, the proof is analogues for $\mathbf{\Delta_{pf}}$. According to Lemma 5.4, $\mathbb{E}\{\mathbf{Q_{pp}}\eta\Pi_p\},\mathbb{E}\{\mathbf{Q_{pp}}\}$ and $\mathbb{E}\{\eta\mathbf{Q_{pp}}W_p^*(W_pW_p^*)^{-2}W_p\}$ are diagonal, from this and

(5.38) we conclude that $\mathbf{\Delta_{pp}}$ is also diagonal, so it is sufficient to evaluate only diagonal terms. We start with $(T_1^{\mathcal{E}})_{rr}$ and use Schwartz inequality :

$$\begin{aligned}|(T_1^{\mathcal{E}})_{rr}| &= \left| \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\mathbf{Q}_{\mathbf{pp}}^{\circ} J_N^{(l)} \right)_{rr} \frac{1}{N} \mathrm{Tr}\left((I_M \otimes J_M^{(l)}) \eta (W_p W_p^*)^{-1} \right) \right\} \right| \\ &\leq \sum_{l=-(L-1)}^{L-1} \left(\mathbf{Var}\left(\left(\mathbf{Q}_{\mathbf{pp}} J_N^{(l)} \right)_{rr} \right) \mathbf{Var}\left(\frac{1}{N} \mathrm{Tr}\left((I_M \otimes J_M^{(l)}) \eta (W_p W_p^*)^{-1} \right) \right) \right)^{1/2}\end{aligned}$$

We apply (5.78) for $a_1 = \mathbf{e}_r$ and $a_2 = J_N^{(l)} \mathbf{e}_r$ and take into account that $\mathbf{Var}(\frac{1}{N}\mathrm{Tr}((I_M \otimes J_M^{(l)})\eta(W_p W_p^*)^{-1})) = \mathcal{O}(N^{-2})$. Then

$$|(T_1^{\mathcal{E}})_{rr}| \leq \mathcal{O}_z^2 \left(\frac{1}{N^{3/2}} \right) \tag{5.84}$$

For second part we have

$$\begin{aligned}|(T_2^{\mathcal{E}})_{rr}| &= \left| \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\eta \mathbf{Q}_{\mathbf{pp}} J_N^{(l)} \Pi_p \right)_{rr}^{\circ} \frac{1}{N} \mathrm{Tr}\left((I_M \otimes J_M^{(l)}) \eta (W_p W_p^*)^{-1} \right) \right\} \right| \\ &\leq \sum_{l=-(L-1)}^{L-1} \left(\mathbf{Var}\left(\left(\eta \mathbf{Q}_{\mathbf{pp}} J_N^{(l)} \Pi_p \right)_{rr} \right) \mathbf{Var}\left(\frac{1}{N} \mathrm{Tr}\left((I_M \otimes J_M^{(l)}) \eta (W_p W_p^*)^{-1} \right) \right) \right)^{1/2}\end{aligned}$$

From (5.79) we get immediately

$$|(T_2^{\mathcal{E}})_{rr}| = \mathcal{O}_z^2 \left(\frac{1}{N^{3/2}} \right) \tag{5.85}$$

For $T_3^{\mathcal{E}}$ we obtain

$$\begin{aligned}|(T_3^{\mathcal{E}})_{rr}| &= \left| \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\eta \mathbf{Q}_{\mathbf{pp}} W_p^* (W_p W_p^*)^{-1} (I_M \otimes J_M^{(l)}) (W_p W_p^*)^{-1} W_p \right)_{rr}^{\circ} \frac{1}{N} \mathrm{Tr}\left(\eta J_N^{(l)} \Pi_p^{\perp} \mathbf{Q}_{\mathbf{fp}} \right) \right\} \right. \\ &\quad \left. + \sum_{l=-(L-1)}^{L-1} \mathbb{E}\left\{ \left(\eta \mathbf{Q}_{\mathbf{pf}} W_f^* (W_f W_f^*)^{-1} (I_M \otimes J_M^{(l)}) (W_p W_p^*)^{-1} W_p \right)_{rr}^{\circ} \frac{1}{N} \mathrm{Tr}\left(\eta J_N^{(l-L)} \Pi_f^{\perp} \mathbf{Q}_{\mathbf{pp}} \right) \right\} \right| \\ &\leq \sum_{l=-(L-1)}^{L-1} \left(\mathbf{Var}\left(\left(\eta \mathbf{Q}_{\mathbf{pp}} W_p^* (W_p W_p^*)^{-1} (I_M \otimes J_M^{(l)}) (W_p W_p^*)^{-1} W_p \right)_{rr} \right) \mathbf{Var}\left(\frac{1}{N} \mathrm{Tr}\left(\eta J_N^{(l)} \Pi_p^{\perp} \mathbf{Q}_{\mathbf{fp}} \right) \right) \right)^{1/2} \\ &\quad + \sum_{l=-(L-1)}^{L-1} \left(\mathbf{Var}\left(\left(\eta \mathbf{Q}_{\mathbf{pf}} W_f^* (W_f W_f^*)^{-1} (I_M \otimes J_M^{(l)}) (W_p W_p^*)^{-1} W_p \right)_{rr} \right) \mathbf{Var}\left(\frac{1}{N} \mathrm{Tr}\left(\eta J_N^{(l)} \Pi_f^{\perp} \mathbf{Q}_{\mathbf{pp}} \right) \right) \right)^{1/2}\end{aligned}$$

from what, using again (5.77) and (5.76) we immediately get

$$|(T_3^{\mathcal{E}})_{rr}| = \mathcal{O}_z^2 \left(\frac{1}{N^{3/2}} \right) \tag{5.86}$$

Finally, we have to deal with part $\mathbb{E}\{\mathcal{E}\}$, for this we remind that each of its terms is of the form $\mathbb{E}\{\frac{\partial \eta}{\partial W_{i_3 j_3}^{m_3}} F\}$, where F is some, maybe random, bound factor. Schwartz inequality and Lemma 5.2 gives us

$$\mathbb{E}\left\{ \frac{\partial \eta}{\partial W_{i_3 j_3}^{m_3}} F \right\} \leq \mathbb{E}\left\{ \left| \frac{\partial \eta}{\partial W_{i_3 j_3}^{m_3}} \right|^2 \right\}^{1/2} \mathbb{E}\{F^2\}^{1/2} = \mathcal{O}_z^2 \left(\frac{1}{N^k} \right)$$

for any $k \in \mathbb{N}$. Since the number of such terms in $\mathbb{E}\{\mathcal{E}\}$ is the multiply of L, we still have $\mathbb{E}\{\mathcal{E}\} = \mathcal{O}_z^2(N^{-k})$. Combining all above we conclude (5.40).

To evaluate the normalized trace of $\mathbf{\Delta_{pp}}$ we follow the same steps as above but with only difference that we can use the better estimate of traces (5.74)-(5.75) instead of ones for quadratic forms (5.78)-(5.79) which will allow us to obtain order of $\mathcal{O}_z^2(N^{-2})$. ∎

5.1.5 No eigenvalues outside the support.

We denote corresponding measure of $\tilde{t}_N$ by $\tilde{\nu}_N$ and its support by $\mathcal{S}_N$. The goal of this Section is to prove the next Theorem.

Theorem 5.1. *Assume that there exists $\epsilon > 0$, $\kappa_1 \in \mathbb{R}$, $\kappa_2 \in \mathbb{R} \cup \{+\infty\}$ and an integer N_0 such that*

$$(\kappa_1 - \epsilon,\, \kappa_2 + \epsilon) \cap \mathcal{S}_N = \varnothing \qquad \forall\, N \geq N_0.$$

Then with probability one, no eigenvalues of $\eta_N^2 \Pi_{p,N} \Pi_{f,N}$ appears in $[\kappa_1,\, \kappa_2]$ for all N large enough.

We first remark that we can consider the case where $\kappa_2 < +\infty$. Indeed, we recall that $\cup_{N \geq 1} \mathcal{S}_N$ is a compact subset, and almost surely, the largest eigenvalue of $\eta_N^2 \Pi_{p,N} \Pi_{f,N}$ is for each N large enough upperbounded by 1.
In order to establish Theorem 5.1, we use the Haagerup-Thornbjornsen approach ([17], see also [7]). For this we remark that in Section 5.1.3 we basically proved the next proposition.

Proposition 5.2. $\forall z \in \mathbb{C}^+$, *we have for N large enough,*

$$\mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} Q_N(z) \right\} = \tilde{t}_N + \frac{1}{N^2} r_N(z)$$

where r_N is holomorphic in $\mathbb{C}^+$ and satisfies

$$|r_N(z)| \leq P_1(|z|) P_2\left(\frac{1}{\mathrm{Im} z}\right)$$

for each $z \in \mathbb{C}^+$, where P_1 and P_2 are nice polynomials.

Proof. Due to (5.65) for $z \in \mathbb{C}^+$ we have $\tilde{\alpha}_N(z) - \tilde{t}_N(z) = \mathcal{O}_z(N^{-2})$, so it is sufficient to remark that since $(\rho(z))^{-1} \leq (\mathrm{Im} z)^{-1}$, there exist two nice polynomials P_1 and P_2 such that $\tilde{\alpha}_N(z) - \tilde{t}_N(z) \leq P_1(|z|) P_2(\frac{1}{\mathrm{Im} z})$.

We now follow [8] and [17] and use the following Lemma

Lemma 5.8. *Let ϕ be a compactly supported real valued smooth function defined on $\mathbb{R}^+$, i. e. $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^+, \mathbb{R}^+)$. Then,*

$$\mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} \phi(\eta_N^2 \Pi_p \Pi_f) \right\} - \int_{\mathcal{S}_N} \phi(\lambda) d\tilde{\nu}_N(\lambda) = \mathcal{O}\left(\frac{1}{N^2}\right) \tag{5.87}$$

Proof. Due to Proposition 2.1 we can write

$$\mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} \phi(\eta_N^2 \Pi_p \Pi_f) \right\} = \frac{1}{\pi} \lim_{y \downarrow 0} \mathrm{Im} \left\{ \int_{\mathbb{R}^+} \phi(x) \mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} Q(x+iy) \right\} dx \right\} \tag{5.88}$$

as well as

$$\int_{\mathcal{S}_N} \phi(\lambda) d\tilde{\nu}_N(\lambda) = \frac{1}{\pi} \lim_{y \downarrow 0} \mathrm{Im} \left\{ \int_{\mathbb{R}^+} \phi(x) \tilde{s}_N(x+iy) dx \right\} \tag{5.89}$$

Using Proposition 5.2, we obtain

$$\mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} \phi(\eta_N^2 \Pi_p \Pi_f) \right\} - \int_{\mathcal{S}_N} \phi(\lambda) d\tilde{\nu}_N(\lambda) = \frac{1}{N^2} \frac{1}{\pi} \lim_{y \downarrow 0} \mathrm{Im} \left\{ \int_{\mathbb{R}^+} \phi(x) r_N(x+iy) dx \right\} \tag{5.90}$$

Since the function $r_N(z) = \mathcal{O}_z(1)$, we can use the result which was proved in [7, Section 3.3] and obtain

$$\limsup_{y \downarrow 0} \left| \int_{\mathbb{R}^+} \phi(x) r_N(x+iy) dx \right| \leq \kappa, \tag{5.91}$$

for some nice constant κ. This and (5.90) complete the proof. ■

In order to establish Theorem 5.1, we introduce a function $\phi \in \mathcal{C}_c^\infty$ such that $0 \le \phi(\lambda) \le 1$ and

$$\phi_0(\lambda) = \begin{cases} 1, \text{ for } \lambda \in [\kappa_1, \kappa_2], \\ 0, \text{ for } \lambda \in \mathbb{R} - (\kappa_1 - \epsilon, \kappa_2 + \epsilon) \end{cases}$$

Since for N large enough $(\kappa_1 - \epsilon, \kappa_2 + \epsilon) \cap \mathcal{S}_N = \varnothing$ then $\int_{\mathcal{S}_N} \phi(\lambda) d\tilde{\nu}_N(\lambda) = 0$ and according to Lemma 5.8

$$\mathbb{E}\left\{ \frac{1}{N} \mathrm{Tr} \phi_0(\eta_N^2 \Pi_p \Pi_f) \right\} = \mathcal{O}\left(\frac{1}{N^2} \right).$$

Now we show that

$$\mathbf{Var}\left\{ \frac{1}{N} \mathrm{Tr} \phi_0(\eta_N^2 \Pi_p \Pi_f) \right\} = \mathcal{O}\left(\frac{1}{N^4} \right)$$

For this we use again the Poincaré-Nash inequality

$$\begin{aligned} \mathbf{Var}\{\mathrm{Tr}\phi_0(\eta_N^2 \Pi_p \Pi_f)\} \le \sum \mathbb{E}\left\{ \frac{\partial \mathrm{Tr}\phi_0(\eta_N^2 \Pi_p \Pi_f)}{\partial W_{i_1,j_1}^{m_1}} \mathbb{E}\{W_{i_1,j_1}^{m_1} \overline{W}_{i_2,j_2}^{m_2}\} \left(\frac{\partial \mathrm{Tr}\phi(\eta_N^2 \Pi_p \Pi_f)}{\partial W_{i_2,j_2}^{m_2}} \right)^* \right\} \\ + \sum \mathbb{E}\left\{ \left(\frac{\partial \mathrm{Tr}\phi_0(\eta_N^2 \Pi_p \Pi_f)}{\partial \overline{W}_{i_1,j_1}^{m_1}} \right)^* \mathbb{E}\{W_{i_1,j_1}^{m_1} \overline{W}_{i_2,j_2}^{m_2}\} \frac{\partial \mathrm{Tr}\phi(\eta_N^2 \Pi_p \Pi_f)}{\partial \overline{W}_{i_2,j_2}^{m_2}} \right\} \end{aligned} \tag{5.92}$$

We only evaluate the first term of the r.h.s. of the inequality, denoted by ψ, because the second is similar. For this we write first

$$\begin{aligned} \frac{\partial \mathrm{Tr}\phi_0(\eta_N^2 \Pi_p \Pi_f)}{\partial \overline{W}_{i_1,j_1}^{m_1}} &= \mathrm{Tr}\left(\phi_0'(\eta_N^2 \Pi_p \Pi_f) \frac{\partial (\eta_N^2 \Pi_p \Pi_f)}{\partial \overline{W}_{i_1,j_1}^{m_1}} \right) \\ &= \begin{cases} 1 \le i_1 \le L, \ (\eta_N^2 \Pi_p^\perp \Pi_f \phi_0'(\eta_N^2 \Pi_p \Pi_f) W_p^* (W_p W_p^*)^{-1})_{i_1 j_1}^{m_1} + \mathcal{O}\left(\frac{1}{N^k} \right), \\ L+1 \le i_1 \le 2L, \ (\eta_N^2 \Pi_f^\perp \phi_0'(\eta_N^2 \Pi_p \Pi_f) \Pi_p W_f^* (W_f W_f^*)^{-1})_{(i_1 - L) j_1}^{m_1} + \mathcal{O}\left(\frac{1}{N^k} \right) \end{cases} \end{aligned}$$

for each $k \in \mathbb{N}$, here term $\mathcal{O}(N^{-k})$ represents the term with $\frac{\partial \eta_N}{\partial \overline{W}_{i_1 j_1}^{m_1}}$ in what follow we will omit it, since as we see above it will not give impact. For convenience we denote $A = \eta_N^2 \Pi_p^\perp \Pi_f \phi_0'(\eta_N^2 \Pi_p \Pi_f) W_p^* (W_p W_p^*)^{-1}$ and $B = \eta_N^2 \Pi_f^\perp \phi_0'(\eta_N^2 \Pi_p \Pi_f) \Pi_p W_f^* (W_f W_f^*)^{-1}$, than one can easily see that

$$\begin{aligned} \psi = \frac{1}{N} \sum_{i_1, i_2 = 1}^{L} \sum_{\substack{j_1, m_1 \\ j_2, m_2}} \mathbb{E}\{ A_{i_1 j_1}^{m_1} \delta_{m_1, m_2} \delta_{i_1 + j_1, i_2 + j_2} A_{i_2 j_2}^{*m_2} + A_{i_1 j_1}^{m_1} \delta_{m_1, m_2} \delta_{i_1 + j_1, i_2 + L + j_2} B_{i_2 j_2}^{*m_2} \\ + B_{i_1 j_1}^{m_1} \delta_{m_1, m_2} \delta_{i_1 + L + j_1, i_2 + j_2} A_{i_2 j_2}^{*m_2} + B_{i_1 j_1}^{m_1} \delta_{m_1, m_2} \delta_{i_1 + j_1, i_2 + j_2} B_{i_2 j_2}^{*m_2} \} \end{aligned}$$

For all four terms we can apply the same trick from above, defining $l = i_1 - i_2$ than we have

$$\begin{aligned} \psi = \frac{1}{N} \sum_{l = -(L-1)}^{L-1} \mathbb{E}\Big\{ \mathrm{Tr}(A J_N^{(l)} A^* (J_L^{(l)} \otimes I_M)) + \mathrm{Tr}(A J_N^{(l-L)} B^* (J_L^{(l)} \otimes I_M)) \\ + \mathrm{Tr}(B J_N^{(l+L)} A^* (J_L^{(l)} \otimes I_M)) + \mathrm{Tr}(B J_N^{(l)} B^* (J_L^{(l)} \otimes I_M)) \Big\} \end{aligned}$$

Using (5.82) for respective A, B we obtain

$$|\psi| \le \frac{4(2L-1)}{N} \mathbb{E}\{\mathrm{Tr} A A^* + \mathrm{Tr} B B^*\} \tag{5.93}$$

Let us evaluate

$$\frac{1}{N}\mathbb{E}\{\mathrm{Tr}AA^*\} = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\eta_N^2\Pi_p^\perp\Pi_f\phi_0'(\eta_N^2\Pi_p\Pi_f)W_p^*(W_pW_p^*)^{-2}W_p\phi_0'(\eta_N^2\Pi_p\Pi_f)\Pi_f\Pi_p^\perp\} \tag{5.94}$$

We denote $\xi(\lambda) = (\phi_0'(\lambda))^2$ and recall that $\Pi_f, \Pi_p^\perp \le I_N$ then

$$\frac{1}{N}\mathbb{E}\{\mathrm{Tr}AA^*\} \le \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\xi(\eta_N^2\Pi_p\Pi_f)\eta_N^2W_p^*(W_pW_p^*)^{-2}W_p\}$$

It is easy to see that $\|\eta_N^2W_p^*(W_pW_p^*)^{-2}W_p\| = \|\eta_N^2W_pW_p^*(W_pW_p^*)^{-2}\| = \|\eta_N^2(W_pW_p^*)^{-1}\| \le ((1+\sqrt{c_*})^2 - 2\epsilon)^{-1}$. This allows us to write that $N^{-1}\mathbb{E}\{\mathrm{Tr}AA^*\} \le \kappa N^{-1}\mathbb{E}\{\mathrm{Tr}\xi(\eta_N^2\Pi_p\Pi_f)\}$. Finally Lemma 5.8 implies that $\frac{1}{N}\mathbb{E}\{\mathrm{Tr}(\xi(\eta_N^2\Pi_p\Pi_f)\} = \mathcal{O}(N^{-2})$. As for the second term, $\mathrm{Tr}BB^*$, we will have the same result :

$$\frac{1}{N}\mathbb{E}\{\mathrm{Tr}BB^*\} = \frac{1}{N}\mathbb{E}\{\mathrm{Tr}\eta_N^2\Pi_f^\perp\phi_0'(\eta_N^2\Pi_p\Pi_f)\Pi_pW_f^*(W_fW_f^*)^{-2}W_f\Pi_p\phi_0'(\eta_N^2\Pi_p\Pi_f)\Pi_f^\perp\}$$
$$\le \frac{\kappa}{N}\mathbb{E}\{\mathrm{Tr}\xi(\eta_N^2\Pi_p\Pi_f)\} = \mathcal{O}(N^{-2})$$

Therefore, we have checked that

$$\mathbf{Var}\{\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f)\} = \mathcal{O}\left(\frac{1}{N^2}\right). \tag{5.95}$$

Now we can complete the proof of Theorem 5.1 as in [8]. For this we apply the classical Markov inequality and combine with what above

$$\mathbf{P}\left\{\frac{1}{N}\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f) > \frac{1}{N^{4/3}}\right\} \le N^{8/3}\mathbb{E}\left\{\left(\frac{1}{N}\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f)\right)^2\right\}$$
$$= N^{8/3}\left(\mathbf{Var}\left\{\frac{1}{N}\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f)\right\} + \left(\mathbb{E}\left\{\frac{1}{N}\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f)\right\}\right)^2\right) = \mathcal{O}\left(\frac{1}{N^{4/3}}\right).$$

Applying Borel-Cantelli lemma, for N large enough, we have with probability one

$$\frac{1}{N}\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f) \le \frac{1}{N^{4/3}}$$

By the very definition of function ϕ_0, the number of eigenvalues of matrix $\eta_N^2\Pi_p\Pi_f$ lying in the interval $[\kappa_1, \kappa_2]$ is upper bounded by $\mathrm{Tr}\phi_0(\eta_N^2\Pi_p\Pi_f) \le \frac{1}{N^{1/3}}$. Since this number of eigenvalues is an integer, we conclude that with probability one there is no eigenvalues in the interval $[\kappa_1, \kappa_2]$ for each N large enough.

5.2 In the presence of signal

In this section we assume that signal $(u_n)_{n\in\mathbb{Z}}$ is present, and evaluate its influence on the eigenvalues of matrix $\Pi_{p,N}\Pi_{f,N}$. For this, we notice that matrices $Y_p^*(Y_pY_p^*)^{-1}Y_p$ and $Y_f^*(Y_fY_f^*)^{-1}Y_f$ are finite rank perturbation of matrices $W_p^*(W_pW_p^*)^{-1}W_p$ and $W_f^*(W_fW_f^*)^{-1}W_f$ due to the noise $(v_n)_{n\in\mathbb{Z}}$, so we can use the same approach as in the previous chapter. Since the useful signal $(u_n)_{n\in\mathbb{Z}}$ is generated by the same minimal state-space representation (1.2), we are keeping notations from the Section 4.1. So as before we denote $\Sigma_{i,N} = \frac{Y_{i,N}}{\sqrt{N}} = W_{i,N} + \Theta_{i,N}\Delta_{i,N}\tilde{\Theta}_{i,N}$ and $\Pi_{i,N}^W = W_i^*(W_iW_i^*)^{-1}W_i$ for $i = p, f$. Let us remind that in the presence of signal we can not assume that $R_N = I_M$, thus $W_i = (I_L \otimes R_N)^{1/2}W_{i,iid}$.
Also, naturally, we keep assumptions related to the signal model (Assumptions 2 and 3), as well as Assumption 6 on the limits of Δ_N and Γ_N. However, it appears that assumptions related to the asymptotic behaviour of the eigenvalue distribution of R_N (Assumption 4, 5) as well as assumption related to the asymptotic behaviour of matrix which depends on both, signal and noise (Assumption 7), are not needed here and can be replaced with milder one.

Assumption 1. *$r \times r$ matrix $G_N = \Theta_N^*(I_L \otimes R_N^{-1})\Theta_N$ converge towards some matrix G_*.*

In order to characterize the possible eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ that escape from $\mathcal{S}_* = [0, 4c_*(1-c_*)] \cup \{1\}\mathbf{1}_{c_*>1/2}$ we, as before, consider the squares of the positive eigenvalues of linearised version, i.e. $\begin{pmatrix} 0 & \Pi_p \\ \Pi_f & 0 \end{pmatrix}$, that escape form $[0, 2\sqrt{c_*(1-c_*)}] \cup \{1\}\mathbf{1}_{c_*>1/2}$. For this we should first find the convenient expression for $\Pi_i - \Pi_i^W$.

Since it will be analogous for $i = p, f$, we consider only Π_p and for simplicity we drop index p. It is easy to see that $\Sigma\Sigma^*$ can be expressed as

$$\Sigma\Sigma^* = WW^* + (W\tilde{\Theta}\Delta, \Theta)\begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix}\begin{pmatrix} \Delta\tilde{\Theta}^*W^* \\ \Theta^* \end{pmatrix}$$

Then with help of Woodbury identity we take the inverse od both sides :

$$(\Sigma\Sigma^*)^{-1} = (WW^*)^{-1} - ((WW^*)^{-1}W\tilde{\Theta}\Delta, (WW^*)^{-1}\Theta)D\begin{pmatrix} \Delta\tilde{\Theta}^*W^*(WW^*)^{-1} \\ \Theta^*(WW^*)^{-1} \end{pmatrix},$$

where

$$D = \left(I_{2r} + \begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix}\begin{pmatrix} \Delta\tilde{\Theta}^*\Pi^W\tilde{\Theta}\Delta & \Delta\tilde{\Theta}^*W^*(WW^*)^{-1}\Theta \\ \Theta^*(WW^*)^{-1}W\tilde{\Theta}\Delta & \Theta^*(WW^*)^{-1}\Theta \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix} \tag{5.96}$$

In order to avoid heavy calculations in what follows, below we simplify separately two expressions

$$\begin{aligned}(\Theta^*(WW^*)^{-1}W\tilde{\Theta}\Delta, \Theta^*(WW^*)^{-1}\Theta)D &= (I_r, 0)\begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix}\begin{pmatrix} \Delta\tilde{\Theta}^*\Pi^W\tilde{\Theta}\Delta & \Delta\tilde{\Theta}^*W^*(WW^*)^{-1}\Theta \\ \Theta^*(WW^*)^{-1}W\tilde{\Theta}\Delta & \Theta^*(WW^*)^{-1}\Theta \end{pmatrix}D \\ &= (I_r, 0)\left(\begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix}\begin{pmatrix} \Delta\tilde{\Theta}^*\Pi^W\tilde{\Theta}\Delta & \Delta\tilde{\Theta}^*W^*(WW^*)^{-1}\Theta \\ \Theta^*(WW^*)^{-1}W\tilde{\Theta}\Delta & \Theta^*(WW^*)^{-1}\Theta \end{pmatrix} + I_{2r} - I_{2r}\right)D \\ &= (I_r, 0)\begin{pmatrix} 0 & I_r \\ I_r & \Delta^2 \end{pmatrix} - (I_r, 0)D = (0, I_r) - (I_r, 0)D \end{aligned} \tag{5.97}$$

and analogous

$$D\begin{pmatrix} \Delta\tilde{\Theta}^*W^*(WW^*)^{-1}\Theta \\ \Theta^*(WW^*)^{-1}\Theta \end{pmatrix} = \begin{pmatrix} 0 \\ I_r \end{pmatrix} - D\begin{pmatrix} I_r \\ 0 \end{pmatrix} \tag{5.98}$$

Now, with (5.97), one can easily obtain that

$$\begin{aligned}\Sigma^*(\Sigma\Sigma^*)^{-1} &= W^*(WW^*)^{-1} - (\Pi^W\tilde{\Theta}\Delta, W^*(WW^*)^{-1}\Theta)D\begin{pmatrix} \Delta\tilde{\Theta}^*W^*(WW^*)^{-1} \\ \Theta^*(WW^*)^{-1} \end{pmatrix} \\ &\quad + \tilde{\Theta}\Delta\Theta^*(WW^*)^{-1} - \left((0, \tilde{\Theta}\Delta) - (\tilde{\Theta}\Delta, 0)D\right)\begin{pmatrix} \Delta\tilde{\Theta}^*W^*(WW^*)^{-1} \\ \Theta^*(WW^*)^{-1} \end{pmatrix} \\ &= W^*(WW^*)^{-1} - (-\Pi^{W,\perp}\tilde{\Theta}\Delta, W^*(WW^*)^{-1}\Theta)D\begin{pmatrix} \Delta\tilde{\Theta}^*W^*(WW^*)^{-1} \\ \Theta^*(WW^*)^{-1} \end{pmatrix}\end{aligned}$$

Finally, multiplying both sides of the last equation by Σ and taking into account (5.98) we have

$$\begin{aligned}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma &= W^*(WW^*)^{-1}W - (-\Pi^{W,\perp}\tilde{\Theta}\Delta, W^*(WW^*)^{-1}\Theta)D\begin{pmatrix} \Delta\tilde{\Theta}^*\Pi^W \\ \Theta^*(WW^*)^{-1}W \end{pmatrix} \\ &\quad + W^*(WW^*)^{-1}\Theta\Delta\tilde{\Theta}^* - (-\Pi^{W,\perp}\tilde{\Theta}\Delta, W^*(WW^*)^{-1}\Theta)\left(\begin{pmatrix} 0 \\ \Delta\tilde{\Theta}^* \end{pmatrix} - D\begin{pmatrix} \Delta\tilde{\Theta}^* \\ 0 \end{pmatrix}\right) \\ &= W^*(WW^*)^{-1}W - (-\Pi^{W,\perp}\tilde{\Theta}\Delta, W^*(WW^*)^{-1}\Theta)D\begin{pmatrix} -\tilde{\Delta}\Theta^*\Pi^{W,\perp} \\ \Theta^*(WW^*)^{-1}W \end{pmatrix}\end{aligned}$$

This allows us to conclude that for $i = p, f$

$$\Pi_i - \Pi_i^W = -\mathcal{A}_i D_i \mathcal{A}_i^*,$$

where

$$\mathcal{A}_i = (-\Pi_i^{W,\perp}\tilde{\Theta}_i\Delta_i, W_i^*(W_iW_i^*)^{-1}\Theta_i) \tag{5.99}$$

Now it is easy to check that

$$\begin{pmatrix} -zI_N & \Pi_p \\ \Pi_f & -zI_N \end{pmatrix} = \begin{pmatrix} -zI_N & \Pi_p^W \\ \Pi_f^W & -zI_N \end{pmatrix} - \begin{pmatrix} \mathcal{A}_p & 0 \\ 0 & \mathcal{A}_f \end{pmatrix} \begin{pmatrix} D_p & 0 \\ 0 & D_f \end{pmatrix} \begin{pmatrix} 0 & \mathcal{A}_p^* \\ \mathcal{A}_f^* & 0 \end{pmatrix} \tag{5.100}$$

As for the first model in Chapter 4, we denote by $\mathbf{Q^W}(z)$ the resolvent of $\begin{pmatrix} 0 & \Pi_p^W \\ \Pi_f^W & 0 \end{pmatrix}$ and consider a positive real number y such that y is not an eigenvalue of $\begin{pmatrix} 0 & \Pi_p^W \\ \Pi_f^W & 0 \end{pmatrix}$ for each large enough N. Then with $z = y$ we rewrite r.h.s. of (5.100) as

$$\begin{pmatrix} -yI_N & \Pi_p \\ \Pi_f & -yI_N \end{pmatrix} = \begin{pmatrix} -yI_N & \Pi_p^W \\ \Pi_f^W & -yI_N \end{pmatrix} \left(I_{2N} - \mathbf{Q^W}(y) \begin{pmatrix} \mathcal{A}_p & 0 \\ 0 & \mathcal{A}_f \end{pmatrix} \begin{pmatrix} D_p & 0 \\ 0 & D_f \end{pmatrix} \begin{pmatrix} 0 & \mathcal{A}_p^* \\ \mathcal{A}_f^* & 0 \end{pmatrix} \right) \tag{5.101}$$

This gives us that y is an eigenvalue of $\begin{pmatrix} 0 & \Pi_p \\ \Pi_f & 0 \end{pmatrix}$ if and only if the determinant of the second factor on r.h.s. of (5.101) equals to 0, or equivalently

$$\det \left(I_{2r} - \begin{pmatrix} \mathcal{A}_p^*\mathbf{Q^W_{fp}}(y)\mathcal{A}_p & \mathcal{A}_p^*\mathbf{Q^W_{ff}}(y)\mathcal{A}_f \\ \mathcal{A}_f^*\mathbf{Q^W_{pp}}(y)\mathcal{A}_p & \mathcal{A}_f^*\mathbf{Q^W_{pf}}(y)\mathcal{A}_f \end{pmatrix} \begin{pmatrix} D_p & 0 \\ 0 & D_f \end{pmatrix} \right) = 0 \tag{5.102}$$

Also since due to (5.96) matrices $D_{p,f}$ are invertible, last equation is also equivalent to

$$\det \left(\begin{pmatrix} D_p^{-1} & 0 \\ 0 & D_f^{-1} \end{pmatrix} - \begin{pmatrix} \mathcal{A}_p^*\mathbf{Q^W_{fp}}(y)\mathcal{A}_p & \mathcal{A}_p^*\mathbf{Q^W_{ff}}(y)\mathcal{A}_f \\ \mathcal{A}_f^*\mathbf{Q^W_{pp}}(y)\mathcal{A}_p & \mathcal{A}_f^*\mathbf{Q^W_{pf}}(y)\mathcal{A}_f \end{pmatrix} \right) = 0 \tag{5.103}$$

Lemma 5.9. *For each $z \in \mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$, where $\boldsymbol{\mathcal{S}}_* = (-2\sqrt{c_*(1-c_*)},\, 2\sqrt{c_*(1-c_*)}) \cup \{\pm 1\}\mathbf{1}_{c_*>1/2}$ and $i \neq j \in \{p, f\}$ we have :*

— $D_i^{-1} - \begin{pmatrix} -(1-c_N)\Delta_N^2 & I_r \\ I_r & \frac{1}{1-c_N}\Theta_N^*(I_L \otimes R_N^{-1})\Theta_N \end{pmatrix} \to 0$ *almost surely*

— $\mathcal{A}_i^*\mathbf{Q^W_{ji}}\mathcal{A}_i - \begin{pmatrix} -\dfrac{(1-c_N)(1+z\tilde{\mathbf{t}}_N(z))}{z\tilde{\mathbf{t}}_N(z)+1-c_N}\Delta_N^2 & 0 \\ 0 & \dfrac{1+\tilde{\mathbf{t}}_N(z)z}{c_N(1-c_N)}\Theta_N^*(I_L \otimes R_N^{-1})\Theta_N \end{pmatrix} \to 0$ *almost surely*

— $\mathcal{A}_f^*\mathbf{Q^W_{pp}}\mathcal{A}_p - \begin{pmatrix} -\dfrac{(1-c_N)^2}{z^2\tilde{\mathbf{t}}_N(z)}\Gamma_N & 0 \\ 0 & 0 \end{pmatrix} \to 0$ *almost surely*

— $\mathcal{A}_p^*\mathbf{Q^W_{ff}}\mathcal{A}_f - \begin{pmatrix} -\dfrac{(1-c_N)^2}{z^2\tilde{\mathbf{t}}_N(z)}\Gamma_N^* & 0 \\ 0 & 0 \end{pmatrix} \to 0$ *almost surely*

Moreover, the 3 last convergence items hold uniformly on each compact subset of $\mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$.

Proof. The proof of this Lemma is postponed to the Section 5.2.1.

We remind that $\Theta_N^*(I_L \otimes R_N^{-1})\Theta_N$ is denoted by G_N, then after trivial algebra, Lemma (5.9) implies that asymptotically, for $N \to \infty$, the "limiting form" of Eq. (5.103) is

$$\det \begin{pmatrix} \frac{(1-c_N)c_N}{z\tilde{\mathbf{t}}_N(z)+1-c_N}\Delta_N^2 & I_r & \frac{(1-c_N)^2}{z^2\tilde{\mathbf{t}}_N(z)}\Gamma_N^* & 0 \\ I_r & -\frac{1-c_N+z\tilde{\mathbf{t}}_N(z)}{c_N(1-c_N)}G_N & 0 & 0 \\ \frac{(1-c_N)^2}{z^2\tilde{\mathbf{t}}_N(z)}\Gamma_N & 0 & \frac{(1-c_N)c_N}{z\tilde{\mathbf{t}}_N(z)+1-c_N}\Delta_N^2 & I_r \\ 0 & 0 & I_r & -\frac{1-c_N+z\tilde{\mathbf{t}}_N(z)}{c_N(1-c_N)}G_N \end{pmatrix} = 0 \tag{5.104}$$

Replacing $z\tilde{\mathbf{t}}_N(z)+1-c_N$ by $z^2 c_N \mathbf{t}_N(z)$ (see (5.71)) and taking the limits of various terms when $N \to +\infty$ (due to Assumptions 3, 6, 1), following the classical stability results we can expect that the zeroes y of equation (5.103) tend to the zeroes of limiting equation, i.e.

$$\det \begin{pmatrix} \frac{1-c_*}{y^2\mathbf{t}_*(y)}\Delta_*^2 & I_r & \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_*^* & 0 \\ I_r & -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* & 0 & 0 \\ \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_* & 0 & \frac{1-c_*}{y^2\mathbf{t}_*(y)}\Delta_*^2 & I_r \\ 0 & 0 & I_r & -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* \end{pmatrix} = 0. \tag{5.105}$$

Here $\mathbf{t}_*, \tilde{\mathbf{t}}_*$ are $\mathbf{t}_N$,$\tilde{\mathbf{t}}_N$ with c_N replaced by c_*. Provided we establish that (5.105) has a finite number s of positive solutions, a property established below, the classical stability results of the zeros of holomorphic functions derived in [4] (see also [10]) will imply that almost surely, for N large enough, Eq. (5.103) has exactly s positive solutions in $]2\sqrt{c_*(1-c_*)}, 1[$ that converge towards the s positive solutions of (5.105).

We now study the solutions of (5.105). If we interchange the second and third row blocks and second and third column blocks the determinant will not change and with Schur complement formula the l.h.s. of (5.105) becomes

$$\det \begin{pmatrix} -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* & 0 \\ 0 & -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* \end{pmatrix} \det \left[\begin{pmatrix} \frac{1-c_*}{y^2\mathbf{t}_*(y)}\Delta_*^2 & \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_*^* \\ \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_* & \frac{1-c_*}{y^2\mathbf{t}_*(y)}\Delta_*^2 \end{pmatrix} - \begin{pmatrix} -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* & 0 \\ 0 & -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* \end{pmatrix}^{-1} \right]$$

Taking this into account and since $\det \begin{pmatrix} -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* & 0 \\ 0 & -\frac{y^2\mathbf{t}_*(y)}{1-c_*}G_* \end{pmatrix} \neq 0$, Eq. (5.105) equivalent to

$$\det \begin{pmatrix} \frac{1-c_*}{y^2\mathbf{t}_*(y)}(\Delta_*^2 + G_*^{-1}) & \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_*^* \\ \frac{(1-c_*)^2}{y^2\tilde{\mathbf{t}}_*(y)}\Gamma_* & \frac{1-c_*}{y^2\mathbf{t}_*(y)}(\Delta_*^2 + G_*^{-1}) \end{pmatrix} = 0$$

Finally with another Schur complement formula, since $\det\left(\frac{1-c_*}{y^2\mathbf{t}_*(y)}(\Delta_*^2 + G_*^{-1})\right) \neq 0$ we obtained that y is eigenvalues of $\begin{pmatrix} 0 & \Pi_p \\ \Pi_f & 0 \end{pmatrix}$ if and only if

$$\det\left(\frac{(1-c_*)^2}{y^4\mathbf{t}_*^2(y)}(\Delta_*^2 + G_*^{-1}) - \frac{(1-c_*)^4}{y^4\tilde{\mathbf{t}}_*^2(y)}\Gamma_*^*(\Delta_*^2 + G_*^{-1})^{-1}\Gamma_* \right) = 0,$$

or equivalently,

$$\det\left(\frac{1}{(1-c_*)^2}\frac{\tilde{\mathbf{t}}_*^2(y)}{\mathbf{t}_*^2(y)}-\Gamma_*^*(\Delta_*^2+G_*^{-1})^{-1}\Gamma_*(\Delta_*^2+G_*^{-1})^{-1}\right)=0 \tag{5.106}$$

We remind that the eigenvalues of $\Pi_p\Pi_f$ that escape from the limiting spectrum $\mathcal{S}_*=[0,4c_*(1-c_*)]\cup\{1\}\mathbf{1}_{c>1/2}$ of $\Pi_p^W\Pi_f^W$ tend to the squares of the positive solutions of Eq. (5.106) that escape the limiting spectrum $\boldsymbol{\mathcal{S}}_*=[-2\sqrt{c_*(1-c_*)},2\sqrt{c_*(1-c_*)}]\cup\{\pm1\}\mathbf{1}_{c_*>1/2}$ of $\begin{pmatrix}0 & \Pi_p^W\\ \Pi_f^W & 0\end{pmatrix}$. Since the eigenvalues of $\Pi_p\Pi_f$ do not exceed 1, we conclude that we are interested in the solutions of Eq. (5.106) on the interval $]2\sqrt{c_*(1-c_*)},1[$. For more convenience we replace $\tilde{\mathbf{t}}_*(y)=y\tilde{t}_*(y^2)$ and $\mathbf{t}_*(y)=t_*(y^2)$, then by replacing y^2 by y we have $(\frac{\tilde{\mathbf{t}}_*(y)}{\mathbf{t}_*(y)})^2\rightarrow\frac{y\tilde{t}_*^2(y)}{t_*^2(y)}$ and Eq. (5.106) becomes

$$\det\left(y\left(\frac{\tilde{t}_*(y)}{(1-c_*)t_*(y)}\right)^2-\Gamma_*^*(\Delta_*^2+G_*^{-1})^{-1}\Gamma_*(\Delta_*^2+G_*^{-1})^{-1}\right)=0 \tag{5.107}$$

The eigenvalues of $\Pi_p\Pi_f$ that escape from $\mathcal{S}_*$, i.e. that belong to $(4c_*(1-c_*),1)$, tend to the solutions of (5.107) that belong $(4c_*(1-c_*),1)$. Let us also notice that Γ_* is the limit of $\Delta_N\tilde{\Theta}_{f,N}^*\Theta_{p,N}\Delta_N$. Since, according to Assumption 6, matrix Δ_N tends to Δ_*, we conclude that the sequence of rank P matrices $(\tilde{\Theta}_{f,N}^*\Theta_{p,N})_{N\geq1}$ converges towards a rank P matrix Ω_* verifying $\Gamma_*=\Delta_N\Omega_*\Delta_N$. Taking this into account, as well as commonly used fact that $\det\,(I+AB)=\det\,(I+BA)$, we rewrite (5.107) as

$$\det\left(y\left(\frac{\tilde{t}_*(y)}{(1-c_*)t_*(y)}\right)^2-F_*\right)=0, \tag{5.108}$$

where $F_*=\Omega_*^*(I_r+\Delta_*^{-1}G_*^{-1}\Delta_*^{-1})^{-1}\Omega_*(I_r+\Delta_*^{-1}G_*^{-1}\Delta_*^{-1})^{-1}$. Since Ω_* is rank P matrix, this gives immediately that $\operatorname{rank}F_*=\Omega_*=P$. Therefore, if we show that $y(\frac{\tilde{t}_*(y)}{(1-c_*)t_*(y)})^2$ is increasing function on $[4c_*(1-c_*),1]$ it will mean that there exist at most P solutions of Eq. (5.107). This justifies that the stability results of [4] holds, and as consequence, that at most P eigenvalues of $\Pi_p\Pi_f$ are outside $\mathcal{S}_*$. To prove this it is sufficient to show that $\frac{\tilde{t}_*(y)}{t_*(y)}$ is positive increasing function on $[4c_*(1-c_*),1]$. Indeed, since $\tilde{t}_*(y)=c_*t_*(y)-\frac{1-c_*}{y}$, then

$$\frac{\tilde{t}_*(y)}{t_*(y)}=c_*-\frac{1-c_*}{yt_*(y)}. \tag{5.109}$$

Also, due to (5.66), it is easy to see that $y\tilde{t}_*(y)$ is increasing on $[4c_*(1-c_*),1]$. However, $c_*yt_*(y)=y\tilde{t}_*(y)+(1-c_*)$ from what follows that $yt_*(y)$ is also increasing on $[4c_*(1-c_*),1]$. This along with (5.109) give that $\frac{\tilde{t}_*(y)}{t_*(y)}$ is an increasing function on $[4c_*(1-c_*),1]$. Moreover, using (5.68) we can calculate explicitly the value at $4c_*(1-c_*)$:

$$(yt_*(y))\Big|_{y=4c_*(1-c_*)}=\frac{4c_*(1-c_*)(2c_*-1)}{2c_*(2c_*-1)^2}=\frac{2(1-c_*)}{2c_*-1}\Rightarrow\frac{\tilde{t}_*(y)}{t_*(y)}\Big|_{y=4c_*(1-c_*)}=c_*-\frac{2c_*-1}{2}=\frac{1}{2}$$

This proofs that $\frac{\tilde{t}_*(y)}{t_*(y)}$ is also positive on $[4c_*(1-c_*),1]$. Finally, we conclude that $\frac{y\tilde{t}_*^2(y)}{(1-c_*)^2t_*^2(y)}$ is increasing on $[4c_*(1-c_*),1]$ and

$$\frac{y\tilde{t}_*^2(y)}{(1-c_*)^2t_*^2(y)}\Big|_{y=4c_*(1-c_*)}=\frac{4c_*(1-c_*)}{4(1-c_*)^2}=\frac{c_*}{1-c_*}$$

We remark that if $c_*<1/2$, then due to (5.66), (5.68) we have

$$\frac{\tilde{t}_*(y)}{t_*(y)}\Big|_{y=1}=\lim_{y\to1}\frac{c_*4(1-c_*)^2(1-y)\left(y(2c_*-1)-\sqrt{y(y-4c_*(1-c_*))}\right)}{y(1-y)4c_*(1-c_*)\left(y-2(1-c_*)-\sqrt{y(y-4c_*(1-c_*))}\right)}=1-c_*$$

and for $c_* \geq 1/2$

$$\frac{\tilde{t}_*(y)}{t_*(y)}\Big|_{y=1} = \lim_{y\to 1} \frac{c_*\left(y - 2(1-c_*) + \sqrt{y(y-4c_*(1-c_*))}\right)}{y(2c_*-1) + \sqrt{y(y-4c_*(1-c_*))}} = c_*$$

With this we obtained that on the interval $[4c_*(1-c_*), 1]$ for $c_* < 1/2$ the expression $\dfrac{y\tilde{t}_*^2(y)}{(1-c_*)^2 t_*^2(y)}$ takes all possible values from $\frac{c_*}{1-c_*} < 1$ to 1 and for $c_* \geq 1/2$ value of $\dfrac{y\tilde{t}_*^2(y)}{(1-c_*)^2 t_*^2(y)}$ runs from $\frac{c_*}{1-c_*} > 1$ to $(\frac{c_*}{1-c_*})^2$. Since G_* is positive defined matrix, one can notice that $\|F_*\| < 1$, so we conclude that for $c_* \geq 1/2$ almost surely no eigenvalues of $\Pi_p\Pi_f$ escape from $\mathcal{S}_*$ for N big enough and if $c_* < 1/2$ the number of outliers coincides with the number of eigenvalues of F_* that are bigger than $\frac{c_*}{1-c_*}$.

This implies that P coincides with the number of eigenvalues that escape from $\mathcal{S}_*$ if and only if all the non zero eigenvalues of F_* are bigger than $\frac{c_*}{1-c_*}$. In this case, P can be consistently estimated. In order to interpret the practical significance of the above condition on the eigenvalues of F_*, we observe that for N large enough, the singular values of $\tilde{\Theta}_{f,N}^*\tilde{\Theta}_{p,N}$ coincide with the canonical correlation coefficients between the row spaces of $U_{p,N}$ and $U_{f,N}$ which have the same behaviour than the canonical correlation coefficients between the spaces generated by the components of u_n^L and of u_{n+L}^L. So generally speaking we can say that if the canonical correlation coefficients between the past and the future of u are large enough (thus making the singular values of Ω_* large) and if the r eigenvalues of $R_{u,N}^L$ are also large enough (thus making matrix Δ_*^{-1} small), then all non zero P eigenvalues of F_* will be greater than $\frac{c_*}{1-c_*}$ and the number of outliers of $\Pi_{p,N}\Pi_{f,N}$ is a consistent estimator for P in high-dimensional regime.

Finally, we denote by $y_1 \geq y_2 \ldots \geq y_s$ the s solutions of Eq. (5.107) that are greater that $4c_*(1-c_*)$ and summing the above discussion we conclude by the next Theorem.

Theorem 5.2. *If $c_* \geq 1/2$ for N large enough no eigenvalues of $\Pi_{p,N}\Pi_{f,N}$ lie outside the $\mathcal{S}_*$.*
If $c_* < 1/2$ for N large enough, exactly s largest eigenvalues $\hat{\lambda}_{1,N} \ldots \geq \hat{\lambda}_{s,N}$ of $\Pi_{p,N}\Pi_{f,N}$ escape from $[0, 4c_*(1-c_*)]$ and $\hat{\lambda}_{i,N} \to y_i$ for $i = 1, \ldots, s$.

We illustrate the above discussion by numerical experiments showing that eigenvalues outside the bulk indeed tend to the square of solutions of equation (5.106). We consider a simple case, when $P = 2$, $K = 1$ and A is diagonal with eigenvalues a_1 and a_2. Figures 5.1, 5.2 represent histograms of the eigenvalues of realizations of the matrix $(\hat{R}_{f,y}^L)^{-1/2}\hat{R}_{f|p,y}^L(\hat{R}_{p,y}^L)^{-1}\hat{R}_{f|p,y}^{L*}(\hat{R}_{f,y}^L)^{-1/2}$, as well as the graph of the density of measure $\nu_N = \frac{1}{c_N}\tilde{\nu}_N - \frac{1-c_N}{c_N}\delta_0$ and the values of the square of solutions of equation (5.106).
We take $N = 2000$, $M = 130$ and $L = 4$, so $c_N = 0.26$. The eigenvalues of matrix R_N are defined by $\lambda_{k,N} = 1/2 + \frac{\pi}{4}\cos\left(\frac{\pi(k-1)}{2M}\right)$ for $k = 1, \ldots, M$, that makes matrix R_N to verify $\frac{1}{M}\mathrm{Tr}(R_N) \simeq 1$. Figure 5.1 corresponds to a choice of (a_1, a_2) for which $s = 1$, while $s = 2$ in the context of Figure 5.2.

5.2.1 Proof of Lemma 5.9

Since the calculation are mostly similar to ones from Sections 5.1.2 and 5.1.4 we will present only sketch of the proof.
We start with showing the first item of the Lemma. For this we obtain the expression for D_i^{-1} from (5.96) :

$$D_i^{-1} = \begin{pmatrix} -\Delta_i^2 & I_r \\ I_r & 0 \end{pmatrix} + \begin{pmatrix} \Delta_i\tilde{\Theta}_i^*\Pi_i^W\tilde{\Theta}_i\Delta_i & \Delta_i\tilde{\Theta}_i^*W_i^*(W_iW_i^*)^{-1}\Theta_i \\ \Theta_i^*(W_iW_i^*)^{-1}W_i\tilde{\Theta}_i\Delta_i & \Theta_i^*(W_iW_i^*)^{-1}\Theta_i \end{pmatrix} \tag{5.110}$$

Using the same arguments as in Lemma 5.4 it is easy to obtain that $\mathbb{E}\{W_i^*(W_iW_i^*)^{-1}\} = \mathbb{E}\{(W_iW_i^*)^{-1}W_i\} = 0$ for $i = p, f$. Also we remind that $\mathbb{E}\{\Pi_i^W\} = c_N I_r + \mathcal{O}(N^{-k})$ and $\mathbb{E}\{(W_iW_i^*)^{-1}\} = (1-c_N)^{-1}(I_L \otimes R_N^{-1}) + \mathcal{O}(N^{-k})$ for each $k \in \mathbb{N}$ (see (5.19)-(5.20)). Combining this with Poincaré-Nash inequality we obtain immediately the first item of the Lemma.

Before passing to the second item of Lemma we introduce the next result

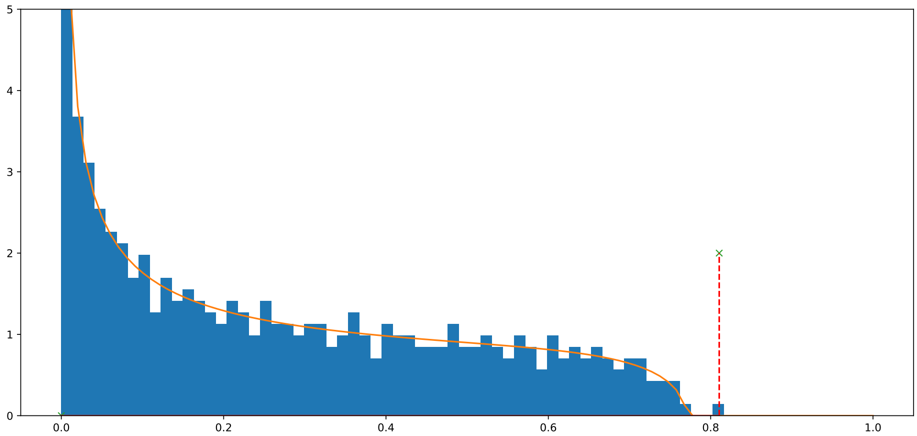

FIGURE 5.1 – Histogram of the eigenvalues and graph of of density with 1 outlier

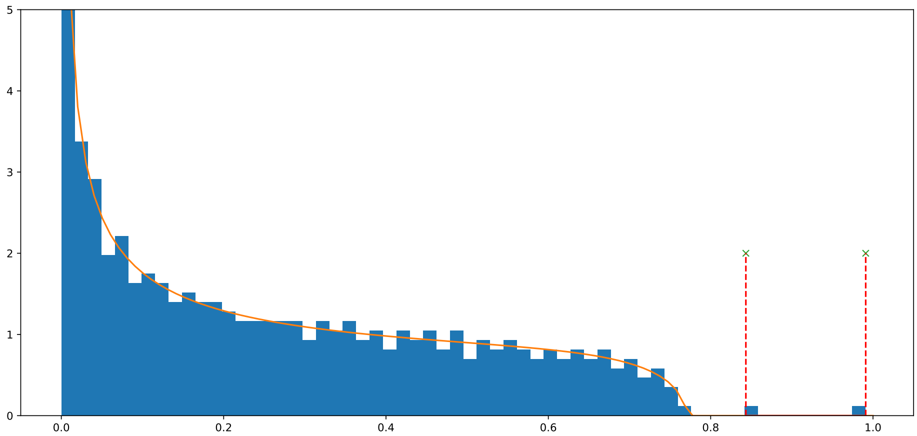

FIGURE 5.2 – Histogram of the eigenvalues and graph of density with 2 outliers

Lemma 5.10. *For each $z \in \mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$, $i \neq j \in \{p, f\}$ and for each bounded sequences $(a_N, b_N)_{N \geq 1}$ of N–dimensional deterministic vectors, it holds that*
— $a_N^* \, \mathbf{Q}^{\mathbf{W}}_{\mathbf{ii}\,N} \, b_N - \tilde{\mathbf{t}}_N(z) a_N^* b_N \to 0$ *almost surely ;*
— $a_N^* \, \mathbf{Q}^{\mathbf{W}}_{\mathbf{ij}\,N} \, b_N - c_N \mathbf{t}_N(z) a_N^* b_N \to 0$ *almost surely.*
Moreover, these converges hold uniformly on each compact subset of $\mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$.

Proof. Due to Remark 5.3, we have that for each z, such that $z^2 \in \mathbb{C}^+$ it holds $\tilde{\boldsymbol{\alpha}}_N(z) - \tilde{\mathbf{t}}_N(z) \to 0$ and $\boldsymbol{\alpha}_N(z) - c_N \mathbf{t}_N(z) \to 0$. Proposition 5.1 implies that both, $\mathbb{E}\{\mathbf{Q_{ii}}_N\}$ and $\mathbb{E}\{\mathbf{Q_{ij}}_N\}$ are multiple of I_N up to an error term, from what immediately we have that each diagonal term of $\mathbb{E}\{\mathbf{Q_{ii}}_N\}$ converges towards $\tilde{\boldsymbol{\alpha}}_N(z)$ and each diagonal term of $\mathbb{E}\{\mathbf{Q_{ij}}_N\}$ converges towards $\boldsymbol{\alpha}_N(z)$. This implies immediately that $a_N^* \mathbb{E}\{\mathbf{Q_{ii}}_N\} b_N - \tilde{\mathbf{t}}_N(z) a_N^* b_N \to 0$ and $a_N^* \mathbb{E}\{\mathbf{Q_{ij}}_N\} b_N - c_N \mathbf{t}_N(z) a_N^* b_N \to 0$ for each z, such that $z^2 \in \mathbb{C}^+$. Using Poincaré-Nash inequality and Borel–Cantelli lemma we obtain almost sure convergence of $a_N^* \, \mathbf{Q_{ii}}_N \, b_N - \tilde{\mathbf{t}}_N(z) a_N^* b_N \to 0$ for each z, such that $z^2 \in \mathbb{C}^+$. To have convergence for each $z \in \mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$ we remark that it can be concluded from Theorem 5.1 that almost surely for each $\delta > 0$ $Q_N(z)$ is analytic on $\mathbb{C} \setminus \mathcal{S}_*^\delta$, where $\mathcal{S}_*^\delta = \{x \in \mathbb{R} : \mathrm{dist}(x, \mathcal{S}_*) \leq \delta\}$. In particular this implies that functions $a_N^* \, \mathbf{Q_{ii}}_N \, b_N$ and $a_N^* \, \mathbf{Q_{ij}}_N \, b_N$ are also analytic on $\mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*^\delta$ for N large enough. The use of Montel's theorem allows to prove the almost sure convergence towards for each $z \in \mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$, as well as the uniformity of the convergence on each compact subset of $\mathbb{C} \setminus \boldsymbol{\mathcal{S}}_*$. ■

We return to the proof of Lemma 5.9. To deal with $\mathcal{A}_i^*\mathbf{Q_{ji}^W}\mathcal{A}_i$ we first notice that similar to what is in Lemma 5.4 it can be proved that $\mathbb{E}\{\Pi_i^W\mathbf{Q_{ji}^W}W_i^*(W_iW_i^*)^{-1}\} = \mathbb{E}\{\mathbf{Q_{ji}^W}W_i^*(W_iW_i^*)^{-1}\} = 0$, from what follows, with Poincaré-Nash inequality, that $r\times r$ blocks on secondary diagonal of $\mathcal{A}_i^*\mathbf{Q_{ji}^W}\mathcal{A}_i$ tend to zeros, i.e. $(\mathcal{A}_i^*\mathbf{Q_{ji}^W}\mathcal{A}_i)(pf)$, $(\mathcal{A}_i^*\mathbf{Q_{ji}^W}\mathcal{A}_i)(fp)\to 0$ (where we refer with $pp, pf, \ldots$ to $r\times r$ blocks of $2r\times 2r$ matrix analogous to (5.13)). Without loss of generality we consider $i=p$, $j=f$. In view of Lemma 5.10, to find an asymptotic equivalent of block $(\mathcal{A}_p^*\mathbf{Q_{fp}^W}\mathcal{A}_p)(pp)$, it is sufficient to express $\Pi_p^{W,\perp}\mathbf{Q_{fp}^W}\Pi_p^{W\perp}$ in terms of $\mathbf{Q_{fp}^W}$ and $\mathbf{Q_{pp}^W}$. Since $\mathbf{Q_{fp}^W} = \Pi_f^W(\Pi_p^W\Pi_f^W - z^2)^{-1} = (\Pi_f^W\Pi_p^W - z^2)^{-1}\Pi_f^W$ (see (5.9)), after straightforward calculations we obtain

$$\Pi_p^{W,\perp}\mathbf{Q_{fp}^W}\Pi_p^{W,\perp} = (\mathbf{Q_{fp}^W} - I_N - z\mathbf{Q_{pp}^W})\Pi_p^{W,\perp} = \mathbf{Q_{fp}^W} - I_N - z\mathbf{Q_{pp}^W} - I_N - z\mathbf{Q_{ff}^W} + \Pi_p^W + z^2\mathbf{Q_{pf}^W}$$
$$\Downarrow$$
$$\mathbb{E}\{\Pi_p^{W,\perp}\mathbf{Q_{fp}^W}\Pi_p^{W,\perp}\} = ((1+z^2)\boldsymbol{\alpha}_N(z) - 1 - 2z\tilde{\boldsymbol{\alpha}}_N(z) - (1-c_N))I_N + \mathcal{O}(N^{-3/2})$$

Due to Lemma 5.10 we have that for any bounded sequences $(a_N, b_N)_{N\geq 1}$ of N–dimensional deterministic vectors $a_N^*\Pi_p^{W,\perp}\mathbf{Q_{fp}}\Pi_p^{W,\perp}b_N - ((1+z^2)c_N\mathbf{t}_N(z) - 1 - 2z\tilde{\mathbf{t}}_N(z) - (1-c_N))a_N^*b_N \to 0$. In order to obtain the corresponding expression stated in the Lemma, we refer to (5.71) and replace $c_N\mathbf{t}_N(z) = \frac{\tilde{\mathbf{t}}_N(z)}{z} + \frac{1-c_N}{z^2}$:

$$(1+z^2)c_N\mathbf{t}_N(z) - 1 - 2z\tilde{\mathbf{t}}_N(z) - (1-c_N) = \tilde{\mathbf{t}}_N(z)\left(\frac{1}{z} - z\right) + \frac{1-c_N}{z^2} - 1 \tag{5.111}$$

Let us remind that $\tilde{\mathbf{t}}_N$ satisfies Eq. (5.53) but in which term $\mathcal{O}_{z^2}(N^{-2})$ is replaced with 0, i.e.

$$(1-z^2)\tilde{\mathbf{t}}_N^2(z) + \left(\frac{2(1-c_N)}{z} - z\right)\tilde{\mathbf{t}}_N(z) + \frac{(1-c_N)^2}{z^2} = 0 \tag{5.112}$$

In order to simplify (5.111) we rewrite Eq. (5.112) as

$$(z\tilde{\mathbf{t}}_N(z) + (1-c_N))\left(\tilde{\mathbf{t}}_N(z)\left(\frac{1}{z} - z\right) + \frac{1-c_N}{z^2} - 1\right) + (1-c_N) + z(1-c_N)\tilde{\mathbf{t}}_N(z) = 0$$

From what we immediately get that r.h.s. of (5.111) equal to $-\frac{(1-c_N)(1+z\tilde{\mathbf{t}}_N(z))}{z\tilde{\mathbf{t}}_N(z)+(1-c_N)}$, what was stated in the Lemma.

Finally, it is left to find an asymptotic of ff block, more precisely, we need to find the asymptotic behaviour of $\mathbb{E}\{(W_pW_p^*)^{-1}W_p\mathbf{Q_{fp}^W}W_p^*(W_pW_p^*)^{-1}\}$. Luckily, due to Lemma 5.4, this matrix is diagonal, so we can consider only the diagonal elements. For this we need to repeat the calculations of Section 5.1.2. In order to avoid another tedious and similar calculation, we provide only the idea and main steps. First it is necessary to apply integration by parts formula for $\sum_{r,t,m_2,i_2}\mathbb{E}\{\mathbf{Q_{fp}^W}_{rt}\bar{W}_{p,i_1t}^{m_1}((W_pW_p^*)^{-1})_{i_1i_2}^{m_1m_2}W_{p,i_2r}^{m_2}\}$ and follow the calculations of Section 5.1.2 applying analogues arguments. In the end we obtain

$$\begin{aligned}\mathbb{E}\{\left((W_pW_p^*)^{-1}W_p\mathbf{Q_{fp}^W}W_p^*\right)_{i_1i_1}^{m_1m_1}\} &= \mathbb{E}\{((W_pW_p^*)^{-1})_{i_1i_1}^{m_1m_1}\}\frac{1}{N}\left(\mathbb{E}\{\mathrm{Tr}\mathbf{Q_{fp}^W}\} - \mathbb{E}\{\mathrm{Tr}\mathbf{Q_{fp}^W}\Pi_p^W\}\right)\\ &- \mathbb{E}\left\{\left((W_pW_p^*)^{-1}W_p\mathbf{Q_{fp}^W}W_p^*(W_pW_p^*)^{-1}\right)_{i_1i_1}^{m_1m_1}\right\}\frac{1}{N}\mathbb{E}\{\mathrm{Tr}\Pi_p^{W,\perp}\mathbf{Q_{fp}^W}\} + \mathcal{O}_{z^2}(N^{-2})\end{aligned}$$

Since $\mathbb{E}\{(W_pW_p^*)^{-1}\} = (1-c_N)^{-1}I_N + \mathcal{O}(N^{-3/2})$ and $\mathbf{Q_{fp}^W}\Pi_p^W = I_N + z\mathbf{Q_{ff}^W}$ we can simplify the r.h.s. of the last equation

$$\begin{aligned}\mathbb{E}\{\left((W_pW_p^*)^{-1}W_p\mathbf{Q_{fp}^W}W_p^*\right)_{i_1i_1}^{m_1m_1}\} &= \frac{1}{1-c_N}(\boldsymbol{\alpha}_N - 1 - z\tilde{\boldsymbol{\alpha}}_N)\\ &- \mathbb{E}\left\{\left((W_pW_p^*)^{-1}W_p\mathbf{Q_{fp}^W}W_p^*(W_pW_p^*)^{-1}\right)_{i_1i_1}^{m_1m_1}\right\}(\boldsymbol{\alpha}_N - 1 - z\tilde{\boldsymbol{\alpha}}_N) + \mathcal{O}_{z^2}(N^{-3/2})\end{aligned} \tag{5.113}$$

Analogous we express $\mathbb{E}\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}W_p^*\big)_{i_1i_1}^{m_1m_1}\}$:

$$\mathbb{E}\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}W_p^*\big)_{i_1i_1}^{m_1m_1}\} = \mathbb{E}\{((W_pW_p^*)^{-1})_{i_1i_1}^{m_1m_1}\}\frac{1}{N}\left(\mathbb{E}\{\mathrm{Tr}\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}\} - \mathbb{E}\{\mathrm{Tr}\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}\Pi_p^W\}\right)$$
$$- \mathbb{E}\left\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}W_p^*(W_pW_p^*)^{-1}\big)_{i_1i_1}^{m_1m_1}\right\}\frac{1}{N}\mathbb{E}\{\mathrm{Tr}\Pi_p^{W,\perp}\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}\} + \mathcal{O}_{z^2}(N^{-2})$$

We remark that $N^{-1}\mathbb{E}\{\mathrm{Tr}\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}\Pi_p^{W,\perp}\} = -z^{-1}\mathbb{E}\{N^{-1}\mathrm{Tr}\Pi_p^{W,\perp}\} = -\frac{1-c_N}{z}$ (see (5.43)), then last equation can be rewritten as

$$\mathbb{E}\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}W_p^*\big)_{i_1i_1}^{m_1m_1}\} = -\frac{1}{z}$$
$$+ \frac{1-c_N}{z}\mathbb{E}\left\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}\big)_{i_1i_1}^{m_1m_1}\right\} + \mathcal{O}_{z^2}(N^{-2}) \quad (5.114)$$

Moreover, using resolvent identity, $(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}W_p^*$ can be rewritten as $(W_pW_p^*)^{-1}W_p(-z^{-1}I_N + z^{-1}(\Pi_f^W\Pi_p^W - z^2)^{-1}\Pi_f^W\Pi_p^W)W_p^* = -z^{-1}I_N + z^{-1}(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*$. Thus, comparing (5.113) and (5.114) we retrieve the necessary formula for diagonal elements of $\mathbb{E}\{(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}\}$:

$$\mathbb{E}\{\big((W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}\big)_{i_1i_1}^{m_1m_1}\} = \frac{\boldsymbol{\alpha}_N - 1 - z\tilde{\boldsymbol{\alpha}}_N}{(1-c_N)((1-c_N)+\boldsymbol{\alpha}_N - 1 - z\tilde{\boldsymbol{\alpha}}_N)} + \mathcal{O}_{z^2}(N^{-2})$$

As we can see, all diagonal elements of $\mathbb{E}\{(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}\}$ are equal up to an error term, what means that the matrix $\mathbb{E}\{(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}\}$ is a multiple of I_N up to an error term. Using again Lemma 5.10, we conclude that

$$a_N^*(W_pW_p^*)^{-1}W_p\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}W_p^*(W_pW_p^*)^{-1}b_N - \frac{c_N\mathbf{t}_N - 1 - z\tilde{\mathbf{t}}_N}{(1-c_N)((1-c_N)+c_N\mathbf{t}_N - 1 - z\tilde{\mathbf{t}}_N)}a_N^*b_N \to 0. \quad (5.115)$$

After replacing $c_N\mathbf{t}_N$ with $\frac{\tilde{\mathbf{t}}_N(z)}{z} + \frac{1-c_N}{z^2}$ we find that $c_N\mathbf{t}_N - 1 - z\tilde{\mathbf{t}}_N = \tilde{\mathbf{t}}_N(z)(\frac{1}{z} - z) + \frac{1-c_N}{z^2} - 1$ which is also the expression obtained in (5.111). Thus, we remind that

$$\tilde{\mathbf{t}}_N(z)\left(\frac{1}{z} - z\right) + \frac{1-c_N}{z^2} - 1 = -\frac{(1-c_N)(1+z\tilde{\mathbf{t}}_N(z))}{z\tilde{\mathbf{t}}_N(z) + (1-c_N)}$$

and by putting this expression in (5.115) we obtain the necessary asymptotic expression for $(\mathcal{A}_p^*\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}\mathcal{A}_p)(ff)$.

Last two items of the Lemma are similar, so we focus only on the one of them, $\mathcal{A}_f^*\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}}\mathcal{A}_p$. Using the similar arguments to ones in Lemma 5.4 we can obtain that pf, fp and ff blocks are zeros. To deal with pp block we remark that $\Pi_f^W\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}} = z\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}}$ and $\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}}\Pi_p^W = z\mathbf{Q}^{\mathbf{W}}_{\mathbf{pf}}$ and express again

$$\Pi_f^{W,\perp}\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}}\Pi_p^{W,\perp} = \mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}} - z\mathbf{Q}^{\mathbf{W}}_{\mathbf{pf}} - z\mathbf{Q}^{\mathbf{W}}_{\mathbf{fp}} + zI_N + z^2\mathbf{Q}^{\mathbf{W}}_{\mathbf{ff}}$$
$$\Downarrow$$
$$\mathbb{E}\{\Pi_f^{W,\perp}\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}}\Pi_p^{W,\perp}\} = \left((1+z^2)\tilde{\boldsymbol{\alpha}}_N(z) - 2z\boldsymbol{\alpha}_N(z) + z\right)I_N + \mathcal{O}_{z^2}(N^{-3/2})$$

Analogous to what above we obtain that $a_N^*\Pi_f^{W,\perp}\mathbf{Q}^{\mathbf{W}}_{\mathbf{pp}}\Pi_p^{W,\perp}b_N - ((1+z^2)\tilde{\mathbf{t}}_N(z) - 2zc_N\mathbf{t}_N(z) + z)a_N^*b_N \to 0$. Using that $c_N\mathbf{t}_N(z) = \frac{\tilde{\mathbf{t}}_N(z)}{z} + \frac{1-c_N}{z^2}$, the limiting expression becomes

$$(1+z^2)\tilde{\mathbf{t}}_N(z) - 2zc_N\mathbf{t}_N(z) + z = (z^2-1)\tilde{\mathbf{t}}_N(z) - \frac{2(1-c_N)}{z} + z$$

With (5.112) we get immediately that r.h.s. of the obtained expression equal to $-\frac{(1-c_N)^2}{z^2\tilde{\mathbf{t}}_N(z)}$. This finishes the proof of the Lemma. ■

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