Outil transformée de Laplace   Signaux causaux

$$X(p) = \underbrace{\int_0^{+\infty} x(t)\, e^{-pt}\, dt}_{\mathcal{L}(x)(p)}$$

$x(t) = e^{at}$ $a > 0$

$X(p)$ bien définie à partir du moment où $\operatorname{Re} p > a$

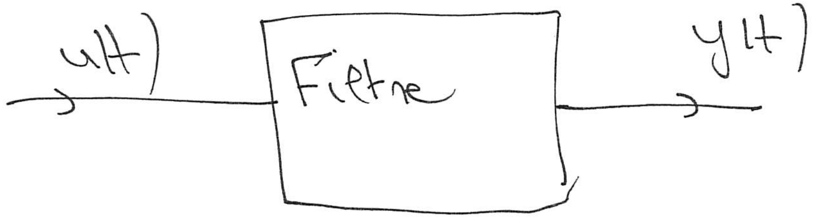
$$\mathcal{L}(x')(p) = p\, X(p)$$

$$y(t) = \int_0^t x(s)\, ds \qquad \mathcal{L}(y) = Y(p) = \frac{X(p)}{p}$$

x, y 2 signaux causaux, $x * y$ signal causal, $(x*y)(t) = 0$, $t < 0$

$$(x * y)(t) = \int_0^t x(s)\, y(t-s)\, ds, \quad \mathcal{L}(x*y)(p) = X(p)\, Y(p)$$

$$\begin{cases} y(t) + \sum_{k=1}^{m} a_k \, y^{(k)}(t) = \sum_{p=0}^{q} b_p \, u^{(p)}(t) \\ u(t) = 0 \text{ si } t<0, \quad y(t) = 0 \text{ si } t<0. \end{cases}$$



$$\begin{cases} y(t) = (h * x)(t) \\ Y(p) = H(p)\,X(p) \end{cases}$$

$$\mathcal{L}\Big(y + \sum_{k=1}^{m} a_k \, y^{(k)}\Big)(p) = \mathcal{L}(y)(p) + \sum_{k=1}^{m} a_k \underbrace{\mathcal{L}(y^{(k)})(p)}$$

$$= Y(p) + \sum_{k=1}^{m} a_k \, p^k \, Y(p)$$

$$= Y(p)\Big(\underbrace{1 + \sum_{k=1}^{m} a_k \, p^k}_{A(p)}\Big) = U(p)\Big(\overbrace{\sum_{p=0}^{q} b_p \, p^p}^{B(p)}\Big)$$

$$Y(p) = H(p)\,U(p), \quad H(p) = \frac{B(p)}{A(p)}$$

$$H(p) = \frac{B(p)}{A(p)} \quad , \quad d^\circ B < d^\circ A$$

$p_1, p_2, \dots, p_m$ les pôles de $H(p)$ $\Leftrightarrow$ les zéros de $A(p)$

$$A(p) = \lambda (p-p_1)(p-p_2)\dots(p-p_m)$$

Si les pôles sont simples $\Leftrightarrow$ $p_1 \neq p_2 \dots \neq p_m$

$$H(p) = \sum_{k=1}^{m} \frac{\lambda_k}{p-p_k}$$

où les λ_k sont des complexes si p_k est complexe, λ_k réel si p_k est réel

$$\frac{1}{p-a} \longmapsto e^{at} \Upsilon(t) \qquad \Upsilon(t) = \mathbb{1}_{\mathbb{R}^+}(t)$$

$$h(t) = \sum_{k=1}^{m} \lambda_k e^{p_k t} \Upsilon(t)$$

En présence d'un (ou plusieurs) pôle complexe, disons p_1

$\Rightarrow$ l'un des autres pôles est p_1^*, disons p_2

$$H(p) = \frac{\lambda_1}{p-p_1} + \frac{\lambda_1^*}{p-p_1^*} + \frac{\lambda_3}{p-p_3} + \dots + \frac{\lambda_n}{p-p_n}$$

$$h(t) = \lambda_1 e^{p_1 t} + \lambda_1^* e^{p_1^* t} + \lambda_3 e^{p_3 t} + \dots + \lambda_n e^{p_n t}$$

$$p_1 = x_1 + i y_1 \quad , \quad x_1 = Re(p_1)$$

$$\lambda_1 = |\lambda_1| e^{i\theta_1} \qquad \lambda_1^* = |\lambda_1| e^{-i\theta_1}$$

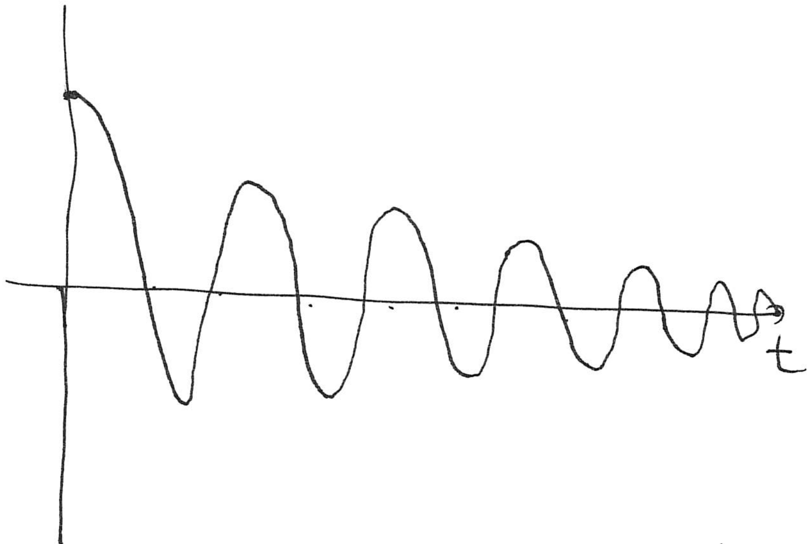
$$\left| \begin{array}{l} e^{p_1 t} = e^{(x_1 + i y_1)t} = e^{x_1 t} e^{i y_1 t} \\ e^{p_1^* t} = e^{(x_1 - i y_1)t} = e^{x_1 t} e^{-i y_1 t} \end{array} \right.$$

$$\lambda_1 e^{p_1 t} + \lambda_1^* e^{p_1^* t}$$

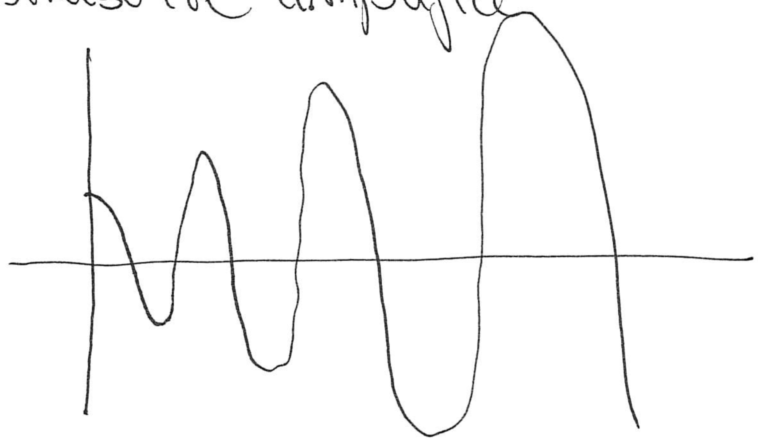
$$= |\lambda_1| e^{i\theta_1} e^{x_1 t} e^{i y_1 t} + |\lambda_1| e^{-i\theta_1} e^{x_1 t} e^{-i y_1 t} = |\lambda_1| e^{x_1 t} \times 2\cos(y_1 t + \theta_1)$$

$2|\lambda_1| e^{x_1 t} \cos(y_1 t + \theta_1)$

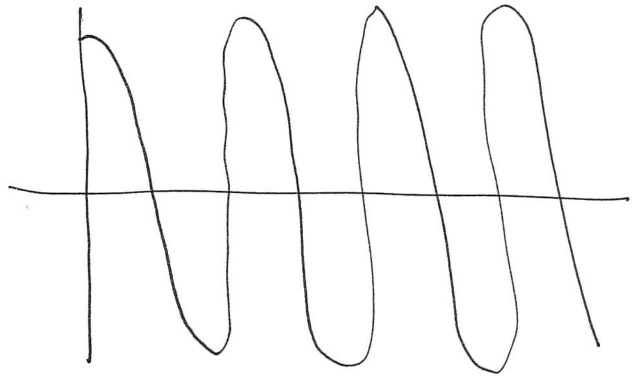
- Si $x_1 < 0$, sinusoïde amortie.



- Si $x_1 > 0$, sinusoïde amplifiée



$x_1 = 0$ $\quad$ $\operatorname{Re} p_1 = 0$, $p_1 = iy_1$



En présence de pôles d'ordre de multiplicité plus grand 1, disons,

$p_{m-1} = p_m$ (multiplicité 2)

$$H(p) = \sum_{k=1}^{m-2} \frac{\lambda_k}{p - p_k} + \frac{\lambda_{m-1}}{(p-p_{m-1})^2} + \frac{\mu_{m-1}}{p - p_{m-1}} \qquad \frac{1}{(p-p_{m-1})^2} \Leftarrow \mathcal{L}\left(t e^{p_{m-1}t}\right)$$

$$h(t) = \sum_{k=1}^{m-2} \lambda_k e^{p_k t} + \underbrace{\mu_{m-1} e^{p_{m-1}t} + \lambda_{m-1} t e^{p_{m-1}t}}_{e^{p_{m-1}t}\left(\mu_{m-1} + \lambda_{m-1} t\right)}$$

Stabilité : $\int_0^{+\infty} |h(t)|\, dt < +\infty$. Stabilité ssi toutes les exponentielles réelles apparaissant dans l'expression de $h(t)$ sont strictement décroissantes. $\Leftrightarrow \forall k=1,\dots,m \quad \mathrm{Re}\, p_k < 0$

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$$y(t) + \tau y'(t) = K u(t)$$

$$\mathcal{L}(y + \tau y')(p) = K U(p)$$

$$\mathcal{L}(y)(p) + \mathcal{L}(\tau y')(p)$$

$$\tau \mathcal{L}(y')(p)$$

$$Y(p) + \tau p Y(p)$$

$$=$$

$$Y(p)(1 + \tau p) = K U(p)$$

$$Y(p) = \frac{K}{1 + \tau p} U(p)$$

$$Y(p) = H(p) U(p)$$

$$H(p) = \frac{K}{1 + \tau p}$$

$H(p)$ a un unique pôle p_1

$$1 + \tau p_1 = 0$$

$$p_1 = -\frac{1}{\tau} < 0$$

Le filtre est stable.

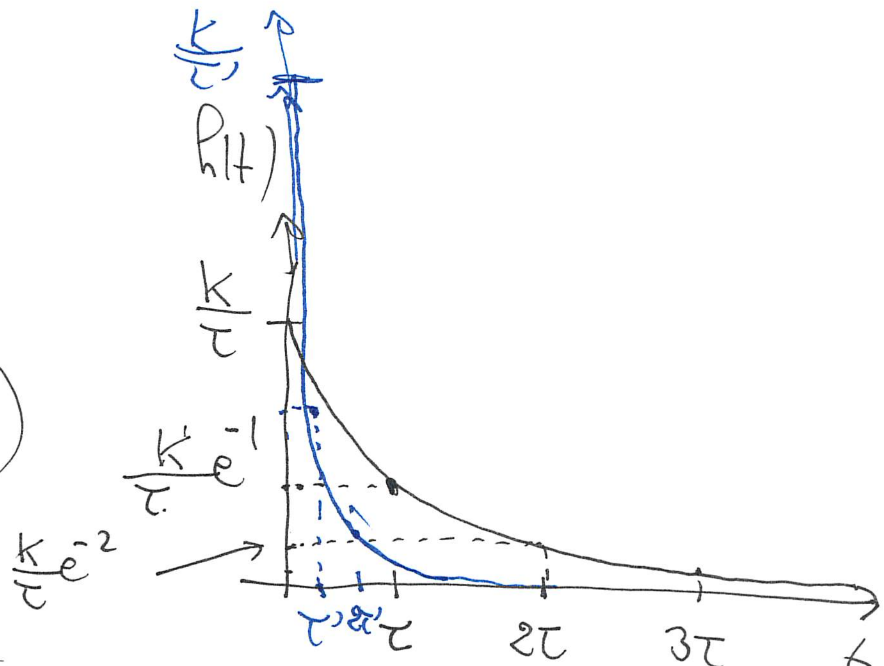
$$\frac{1}{p-a} \longleftrightarrow e^{at}\,\Upsilon(t)$$

$$\frac{K}{1+\tau p} = \frac{K}{\tau\left(p+\frac{1}{\tau}\right)} = \frac{K}{\tau} \cdot \frac{1}{p+\frac{1}{\tau}}$$

$$\frac{1}{p+\frac{1}{\tau}} \longleftrightarrow e^{-t/\tau}\,\Upsilon(t)$$

$$h(t) = K\,\frac{e^{-t/\tau}}{\tau}\,\Upsilon(t)$$

si $t = 2\tau$: $h(2\tau) = \frac{K}{\tau} e^{-2}$



τ = constante de temps

3

$u(t) = \Upsilon(t)$ On veut calculer $y(t)$, appelée réponse indicielle

$$Y(p) = H(p)\,U(p) = \frac{K}{1+\tau p} \cdot \frac{1}{p} = \frac{K}{p(1+\tau p)}$$

Pôles de $Y(p)$: solutions de $p(1+\tau p) = 0$: $p_1 = -\frac{1}{\tau}$, $p_2 = 0$

$$\frac{1}{p(1+\tau p)} = \frac{\lambda}{p-p_1} + \frac{\mu}{p-p_2} = \left[\frac{\mu}{p} + \frac{\lambda}{p+\frac{1}{\tau}} = \frac{1}{p(1+\tau p)}\right.$$

$$p\left(\frac{1}{p(1+\tau p)}\right) = p\left(\frac{\mu}{p} + \frac{\lambda}{p+\frac{1}{\tau}}\right)$$

$$\frac{1}{1+\tau p} = \mu + \frac{\lambda p}{p+\frac{1}{\tau}}\Bigg|_{p=0} \qquad \mu = \left(\frac{1}{1+\tau p}\right)_{p=0} = 1$$