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$$y(t) + \tau y'(t) = K u(t)$$

$\tau > 0$ constante de temps

$K > 0$ gain ~~cons~~ statique

$u(t) = 0 \quad t < 0$

$y(t) = 0 \quad t < 0$

$$Y(p) + \tau p Y(p) = K U(p)$$

$$Y(p) = \frac{K}{1+\tau p} U(p), \qquad H(p) = \frac{K}{1+\tau p}$$

Le filtre est stable, l'unique pôle de $H(p)$ est $p_1 = -\frac{1}{\tau}$

$$\frac{1}{p-a} = \mathcal{L}\left(e^{at} \Upsilon(t) \right)(p)$$

$$\frac{1}{1+\tau p} = \frac{1}{\tau}\left(\frac{1}{p+\frac{1}{\tau}} \right) = \frac{1}{\tau} \mathcal{L}\left(e^{-\frac{1}{\tau} t} \Upsilon(t) \right)(p)$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \Upsilon(t)$$

Réponse indicielle : la sortie du filtre lorsque $u(t) = \Upsilon(t)$ ②

$$y(t) + \tau\, y'(t) = K\, u(t) \iff Y(p)(1+\tau p) = K\, U(p)$$

$$Y(p) = \frac{K}{1+\tau p}\, U(p) = \frac{K}{p(1+\tau p)}$$

$$\frac{1}{p(1+\tau p)} = \frac{1}{p} - \frac{1}{p+\frac{1}{\tau}}$$

Pourquoi ?

$$\frac{1}{p(1+\tau p)} = \frac{\mu}{p} + \frac{\lambda}{p+\frac{1}{\tau}}$$

Voie alternative n°1.

$$\frac{1}{p(1+\tau p)} = \frac{\mu}{p} + \frac{\lambda'}{1+\tau p}$$

$$\mu = 1$$

$$\frac{1}{p} = \cancel{\frac{\mu}{p}}(1+\tau p) + \lambda'$$

$$\lambda' = \left(\frac{1}{p}\right)_{p=-\frac{1}{\tau}} = -\tau$$

$$\frac{1}{p(1+\tau p)} = \frac{1}{p} - \frac{\tau}{1+\tau p} = \frac{1}{p} - \frac{1}{p+\frac{1}{\tau}}$$

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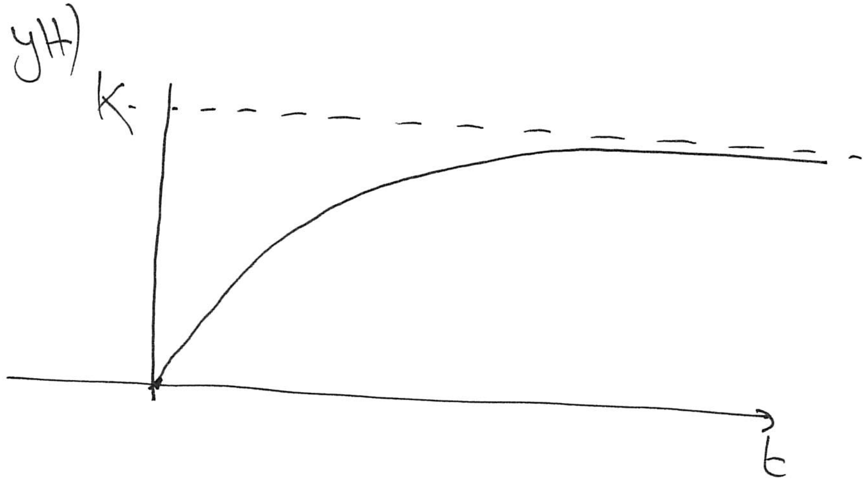
$$\frac{1}{p(1+\tau p)} = \frac{1}{p} - \frac{1}{p+\frac{1}{\tau}}$$

$$Y(p) = \frac{K}{p(1+\tau p)} = K\left(\frac{1}{p} - \frac{1}{p+\frac{1}{\tau}}\right) \Rightarrow y(t) = K\left(\Upsilon(t) - e^{-t/\tau}\Upsilon(t)\right)$$

$$y(t) = K\left(1 - e^{-t/\tau}\right) \quad t \geq 0$$

$$= 0 \quad t < 0$$



$$\lim_{t \to +\infty} y(t) = K$$

$y(t)$ est croissante.

Lorsque on applique au filtre le signal constant égal à 1, la sortie ~~temp~~ tend vers K si $t \to +\infty$.

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$$\frac{K}{1+\tau p} = \mathcal{L}\left(\frac{K}{\tau} e^{-t/\tau} \Gamma(t)\right)(p) = \mathcal{L}\left(h(t)\right)(p)$$

$$\frac{K}{p(1+\tau p)} = \frac{1}{p} \cdot \frac{K}{1+\tau p}$$

$\frac{1}{p} \longleftrightarrow$ l'intégration entre 0 et t.

$$\frac{1}{p} \frac{K}{1+\tau p} \longleftrightarrow \int_0^t h(s)\,ds$$

$$t>0 \quad y(t) = \int_0^t h(s)\,ds = \int_0^t \frac{K}{\tau} e^{-s/\tau}\,ds = \frac{K}{\tau} \underbrace{\int_0^t e^{-s/\tau}\,ds}_{-\tau\left(e^{-s/\tau}\right)_0^t}$$

$$= -K\left(e^{-t/\tau} - 1\right) = K\left(1 - e^{-t/\tau}\right)$$

(5)

Calculer $y(t)$ si $u(t) = t\,\Upsilon(t)$ $\left(u(t) = t \geq 0, \ 0 \text{ si } t < 0 \right)$

$Y(p) = H(p)\,U(p)$ $\qquad U(p)$?

$e^{at}\,\Upsilon(t) \longleftrightarrow \frac{1}{p-a}$ $\qquad \left(a = -\frac{1}{\tau} : \quad \frac{1}{p+\frac{1}{\tau}} \longleftrightarrow e^{-\frac{t}{\tau}}\,\Upsilon(t) \right)$

$t\,e^{at}\,\Upsilon(t) \longleftrightarrow \frac{1}{(p-a)^2}$

$\Rightarrow \quad U(p) = \frac{1}{p^2}$

$Y(p) = \frac{K}{p^2(1+\tau p)}$ $\qquad$ 1 pôle simple $p_1 = -\frac{1}{\tau}$

1 pôle double 0

$$\frac{1}{p^2(1+\tau p)} = \frac{\lambda_1}{p} + \frac{\lambda_2}{p^2} + \frac{\mu}{1+\tau p}$$

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$$p^2 \times \frac{1}{p^2(1+\tau p)} = \left(\frac{\lambda_1}{p} + \frac{\lambda_2}{p^2} + \frac{\mu}{1+\tau p} \right) p^2$$

$$\parallel$$

$$\frac{1}{1+\tau p} = \lambda_1 p + \lambda_2 + \frac{\mu p^2}{1+\tau p}$$

Si $p=0$, $\lambda_2 = \left(\frac{1}{1+\tau p} \right)_{p=0} = 1.$

$$\frac{d}{dp}\left(\frac{1}{1+\tau p} \right) = \frac{d}{dp}\left(\lambda_1 p + \lambda_2 + \frac{\mu p^2}{1+\tau p} \right)$$

$$-\frac{\tau}{(1+\tau p)^2} = \lambda_1 + \mu \left(\frac{p^2}{1+\tau p} \right)' = \lambda_1 + \mu \frac{2p(1+\tau p) - p^2 \tau}{(1+\tau p)^2}$$

$\underline{\underline{p=0}}$ $\quad \lambda_1 = \left[-\frac{\tau}{(1+\tau p)^2} \right]_{p=0} = -\tau$

8 bis

$$(1+\tau p)\frac{1}{p^2(1+\tau p)} = \left(\frac{1}{p^2} - \frac{\tau}{p} + \frac{\mu}{1+\tau p}\right)(1+\tau p)$$

$$\frac{1}{p^2} = \underbrace{(1+\tau p)\left(\frac{1}{p^2} - \frac{\tau}{p}\right)}_{0} + \mu$$

On prend $1+\tau p = 0$, $p = -\frac{1}{\tau}$

$$\mu = \left(\frac{1}{p^2}\right)_{p=-\frac{1}{\tau}} = \tau^2$$

$$\frac{1}{p^2(1+\tau p)} = \frac{1}{p^2} - \frac{\tau}{p} + \frac{\tau^2}{1+\tau p}$$

$$\boxed{\frac{\tau^2}{1+\tau p} = \frac{\tau^2}{\tau\left(p+\frac{1}{\tau}\right)} = \frac{\tau}{p+\frac{1}{\tau}}}$$

$$Y(p) = \frac{K}{p^2(1+\tau p)} = \frac{K}{p^2} - \frac{K\tau}{p} + \frac{K\tau^2}{1+\tau p}$$

$$y(t) = K\left[t\Upsilon(t) - \tau\Upsilon(t) + \tau e^{-\frac{t}{\tau}}\Upsilon(t)\right]$$

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$$y(t) = K\left[t - \tau + \tau e^{-t/\tau}\right] \Upsilon(t)$$

$y(t) = 0$ si $t < 0$

$$y(t) = K\left(t - \tau + \tau e^{-t/\tau}\right)$$

Si $u(t) = A \cos 2\pi f_0 t \; \Upsilon(t)$

Question préalable : si $u(t) = A \cos 2\pi f_0 t \quad \forall t$

$$y(t) = A\left|H(2i\pi f_0)\right| \cos\left(2\pi f_0 t + \operatorname{Arg} H(2i\pi f_0)\right)$$

$$H(p) = \frac{K}{1+\tau p}$$

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$u(t) = A \cos 2\pi f_0 t \;\Upsilon(t)$

$y(t)$? $Y(p) = H(p)\,U(p)$ $\omega_0 = 2\pi f_0$

$u(t) = A \cos \omega_0 t \;\Upsilon(t)$

$U(p)$? $\cos \omega_0 t = \dfrac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$

$$u(t) = \frac{A}{2}\left(e^{i\omega_0 t}\,\Upsilon(t) + e^{-i\omega_0 t}\,\Upsilon(t)\right)$$

$e^{at}\,\Upsilon(t) \longleftrightarrow \dfrac{1}{p-a}$ | On prend $a = i\omega_0$: $e^{i\omega_0 t}\,\Upsilon(t) \longleftrightarrow \dfrac{1}{p - i\omega_0}$

$a = -i\omega_0$ $e^{-i\omega_0 t}\,\Upsilon(t) \longleftrightarrow \dfrac{1}{p + i\omega_0}$

$$U(p) = \frac{A}{2}\left(\frac{1}{p - i\omega_0} + \frac{1}{p + i\omega_0}\right) = A\,\frac{p}{(p - i\omega_0)(p + i\omega_0)}$$

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$$Y(p) = H(p)U(p) = \frac{K}{(1+\tau p)} \cdot \frac{A\quad p}{(p-i\omega_0)(p+i\omega_0)}$$

$$(1+\tau p) \times \frac{p}{(1+\tau p)(p-i\omega_0)(p+i\omega_0)} = \left(\frac{\lambda}{1+\tau p} + \frac{\mu}{p-i\omega_0} + \frac{\mu^*}{p+i\omega_0}\right)(1+\tau p)$$

On prend $p = -\frac{1}{\tau}$

$$\left(\frac{p}{(p-i\omega_0)(p+i\omega_0)}\right)_{p=-\frac{1}{\tau}} = \lambda = \frac{-\frac{1}{\tau}}{\left(-\frac{1}{\tau}-i\omega_0\right)\left(-\frac{1}{\tau}+i\omega_0\right)} = -\frac{1}{\tau} \frac{1}{\left|\frac{1}{\tau}+i\omega_0\right|^2}$$

$$= -\frac{1}{\tau} \cdot \frac{1}{\left(\frac{1}{\tau^2}+\omega_0^2\right)}$$

$$= -\frac{1}{\tau} \cdot \frac{\tau^2}{1+\omega_0^2\tau^2}$$

$$= -\frac{\tau}{1+\omega_0^2\tau^2}$$

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$$(p-i\omega_0)\left(\frac{p}{(1+\tau p)(p-i\omega_0)(p+i\omega_0)}\right) = \left(\frac{\lambda}{1+\tau p} + \frac{\mu}{p-i\omega_0} + \frac{\mu^*}{p+i\omega_0}\right)(p-i\omega_0)$$

$p = i\omega_0$

$$\left(\frac{p}{(1+\tau p)(p+i\omega_0)}\right)_{p=i\omega_0} = \mu = \frac{i\omega_0}{(1+i\omega_0\tau)\cdot 2i\omega_0} = \frac{1}{2(1+i\omega_0\tau)}$$

$$\left(\frac{p}{(1+\tau p)(p-i\omega_0)}\right)_{p=-i\omega_0} = \frac{-i\omega_0}{(1-i\omega_0\tau)\times -2i\omega_0} = \frac{1}{2(1-i\omega_0\tau)} = \mu^*$$