Phase Retrieval, MAXCUT and Complex Semidefinite Programming

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Introduction

Focus on the **phase retrieval** problem, i.e.

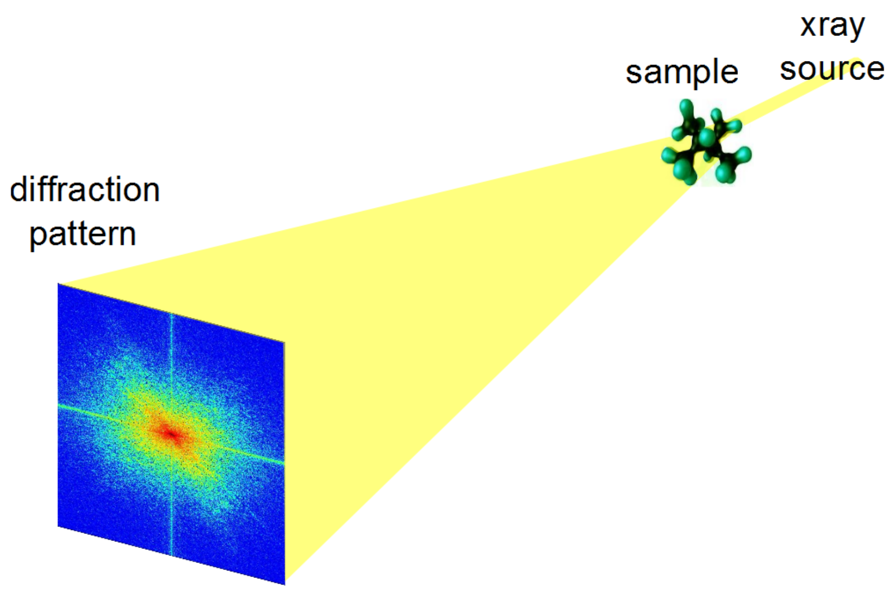
$$\begin{array}{ll} \text{find} & x \\ \text{such that} & |\langle a_i, x\rangle|^2 = b_i^2, \quad i = 1, \ldots, n \end{array}$$

in the variable $x \in \mathbb{C}^p$.

- Reconstruct a signal x from the **amplitude of n linear measurements**.
- We seek a **tractable** procedure, i.e. a polynomial time algorithm with explicit approximation and complexity bounds.

Introduction

Applications in e.g. molecular imaging



(from [Candes et al., 2011b])

- CCD sensors only record the **magnitude** of diffracted rays, and loose the **phase**
- **Fraunhofer diffraction:** phase is required to invert the 2D Fourier transform

Introduction

Problem is almost 100 years old, infinite list of references. . .

Algorithms

- Greedy algorithm [Gerchberg and Saxton, 1972]
- Classical survey of algorithms by [Fienup, 1982].
- NP-complete [Sahinoglou and Cabrera, 1991].
- Matrix completion formulation [Chai, Moscoso, and Papanicolaou, 2011] and [Candes, Strohmer, and Voroninski, 2011a]

Applications

- X-ray and crystallography imaging [Harrison, 1993], diffraction imaging [Bunk et al., 2007] or microscopy [Miao et al., 2008].
- Audio signal processing [Griffin and Lim, 1984].

Introduction

Classical **greedy algorithm** [Gerchberg and Saxton, 1972].

Input: An initial $y^1 \in \mathbb{C}^n$, i.e. such that $|y^1| = b$.

1: **for** $k = 1, \ldots, N-1$ **do**

2: Set

$$w = AA^\dagger y^k$$

3: Set

$$y_i^{k+1} = b_i \frac{w}{|w|}, \quad i = 1, \ldots, n.$$

4: **end for**

Output: $y_N \in \mathbb{C}^n$.

Very similar to **alternating projections**:

- Project on $\mathcal{R}(A)$.
- Adjust the magnitude to match b
- Repeat. . .

Introduction

- [Chai et al., 2011] and [Candes et al., 2011a] use a **lifting** procedure from [Shor, 1987, Lovász and Schrijver, 1991] to write

$$|\langle a_i, x \rangle|^2 = b_i^2 \quad \Longleftrightarrow \quad \mathbf{Tr}(a_i a_i^* x x^*) = b_i^2$$

  and formulate phase recovery as a **matrix completion** problem

$$\begin{array}{ll} \text{Minimize} & \mathbf{Rank}(X) \\ \text{such that} & \mathbf{Tr}(a_i a_i^* X) = b_i^2, \quad i = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

  in the matrix $X \in \mathbf{H}_p$.

- [Recht et al., 2007, Candes and Recht, 2008, Candes and Tao, 2010] show that under certain conditions on A and x_0, it suffices to solve

$$\begin{array}{ll} \text{Minimize} & \mathbf{Tr}(X) \\ \text{such that} & \mathbf{Tr}(a_i a_i^* X) = b_i^2, \quad i = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

  which is a (convex) **semidefinite program** in $X \in \mathbf{H}_p$.

Outline

- Introduction
- **MAXCUT formulation**
- Tightness
- Algorithms & Structure
- Numerical Results

MAXCUT formulation

We can **decouple** the phase and magnitude reconstruction problems.

- In the noiseless case, write $Ax = \mathbf{diag}(b)u$ where $u \in \mathbb{C}^n$ is a **phase vector** with $|u_i| = 1$.
- The phase recovery problem can be written

$$\min_{\substack{u \in \mathbb{C}^n,\, |u_i|=1, \\ x \in \mathbb{C}^p}} \quad \|Ax - \mathbf{diag}(b)u\|_2^2,$$

- The inner minimization problem in x is a standard least squares, with solution $x = A^\dagger \,\mathbf{diag}(b)u$, so phase recovery becomes

$$\begin{array}{ll} \text{minimize} & u^* M u \\ \text{subject to} & |u_i| = 1, \quad i = 1, \ldots n, \end{array}$$

  in $u \in \mathbb{C}^n$, where the Hermitian matrix $M = \mathbf{diag}(b)(\mathbf{I} - AA^\dagger)\,\mathbf{diag}(b)$ is positive semidefinite.

MAXCUT formulation

MAXCUT. Classical algorithm in combinatorial optimization.

- Given an undirected graph with weights w_{ij} on its edges (i,j), *MaxCut* seeks to partition the vertices in two sets S and $\bar{S}$ to **maximize the weight of the cut**

$$\max_{S \subset [1,n]} \sum_{\{i \in S,\ j \in \bar{S}\}} w_{ij}$$

- This can be written as a quadratic program

$$\begin{array}{ll} \text{maximize} & x^T L x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

  where L is the graph Laplacian, $L = \mathbf{diag}(We) - W$.

- Other interpretations as computing the ground state of spin glass models [Mezard and Montanari, 2009], computing mixed matrix norms [Nemirovski, 2005], approximating the CUT-norm [Alon and Naor, 2004], etc...

MAXCUT formulation

MAXCUT. We know a lot about how to find an approximate solution

$$\begin{array}{ll} \text{maximize} & x^T L x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- [Goemans and Williamson, 1995] produce a polynomial algorithm with an approximation ratio of $0.878\ldots$, using a semidefinite relaxation

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}(XL) \\ \text{subject to} & \mathbf{diag}(X) = 1, \; X \succeq 0 \end{array}$$

  combined with a randomization argument.
- Approximating the solution with an approximation ratio better than $16/17$ is NP-Hard, etc.

MAXCUT formulation

The **phase recovery** problem was written (in phase) as

$$\begin{array}{ll} \text{minimize} & u^* M u \\ \text{subject to} & |u_i| = 1, \quad i = 1, \ldots n, \end{array}$$

- We can write a relaxation for phase recovery similar to the MAXCUT SDP, and recycle all the efficient algorithms designed for MAXCUT to solve it.
- Nesterov [1998] produces approximation bounds for generic nonconvex quadratic programs. [Goemans and Williamson, 2001, Zhang and Huang, 2006] extend these results to complex valued problems and show a $\pi/4$ approximation ratio for

  $$\begin{array}{ll} \text{maximize} & u^* M u \\ \text{subject to} & |u_i| = 1, \quad i = 1, \ldots n, \end{array}$$

  when $M \succeq 0$.
- Tightness results on very similar maximum-likelihood channel detection problems [Luo et al., 2003, Kisialiou and Luo, 2010, So, 2010].

Outline

- Introduction
- MAXCUT formulation
- **Tightness**
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Outline

Tightness. [Waldspurger, d'Aspremont, and Mallat, 2012] Write a semidefinite relaxation for phase recovery, similar to the MAXCUT SDP

$$\begin{array}{ll} \text{Minimize} & \mathbf{Tr}(MU) \\ \text{such that} & \mathbf{diag}(U) = 1, \; X \succeq 0 \end{array}$$

call it **PhaseCut**. When do we perfectly recover the signal x?

- [Candes et al., 2011a] show exact recovery w.h.p. for the **PhaseLift** relaxation

$$\begin{array}{ll} \text{Minimize} & \mathbf{Tr}(X) \\ \text{such that} & \mathbf{Tr}(a_i a_i^* X) = b_i^2, \quad i = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

  when $n = O(p \log p)$ observations a_i are picked uniformly on the unit sphere.
- [Waldspurger et al., 2012] show

PhaseCut is tight whenever PhaseLift is.

- Empirically, slightly more robust to noise.

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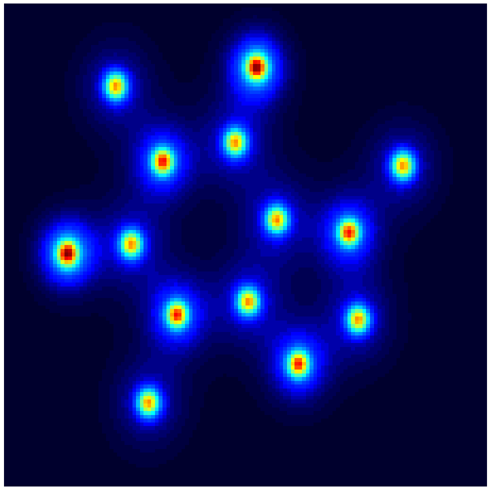
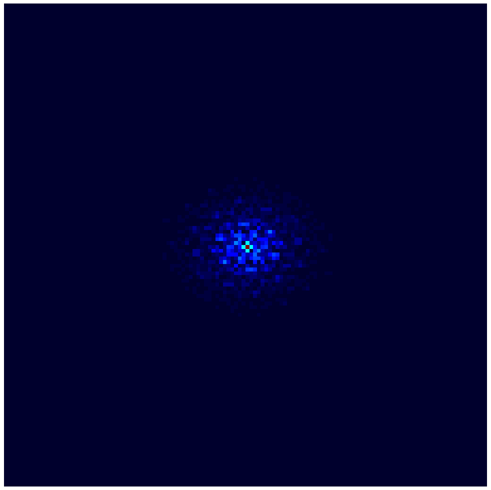
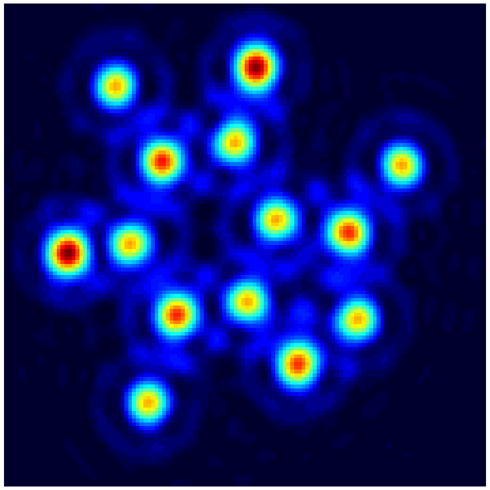
Sparsity: known support in 2D

- Molecular imaging: the samples are approximately sparse with **known support.**
- Most of the coefficient in b are close to zero.







Electronic density for the caffeine molecule (left), its 2D FFT transform (diffraction pattern, center), the density reconstructed using 3% of the coefficients at the core of the FFT (right).

Positivity

- We observe the magnitude of the Fourier transform of a discrete nonnegative signal $x \in \mathbb{R}^p$ so that
$$|\mathcal{F}x| = b$$
- We seek to reconstruct **positive** signals $x \geq 0$.
- This introduces additional **convex** restrictions on the phase vector u.

A function $f : \mathbb{R}^s \mapsto \mathbb{C}$ is *positive semidefinite* if and only if the matrix B with $B_{ij} = f(x_i - x_j)$ is Hermitian positive semidefinite for any sequence $x_i \in \mathbb{R}^s$.

Theorem

Bochner. *A function $f : \mathbb{R}^s \mapsto \mathbb{C}$ is positive semidefinite if and only if it is the Fourier transform of a (finite) nonnegative Borel measure.*

Positivity

- Reconstruct a phase vector $u \in \mathbb{C}^n$ such that $|u| = 1$ and

$$\mathcal{F}x = \mathbf{diag}(b)u.$$

In 1D (for simplicity), if we define the Toeplitz matrix

$$B_{ij}(y) = y_{|i-j|+1}, \quad i,j = 1,\ldots,p,$$

so that

$$B(y) = \begin{pmatrix} y_1 & y_2^* & & \cdots & & y_n^* \\ y_2 & y_1 & y_2^* & & \cdots & \\ & y_2 & y_1 & y_2^* & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & \cdots & & y_2 & y_1 & y_2^* \\ y_n & & \cdots & & y_2 & y_1 \end{pmatrix}$$

- When $\mathcal{F}x = \mathbf{diag}(b)u$, Bochner's theorem means $B(\mathbf{diag}(b)u) \succeq 0$ iff $x \geq 0$.
- The contraint $B(\mathbf{diag}(b)u) \succeq 0$ is a **linear matrix inequality** in u, hence is convex.

Algorithms

PhaseCut is a complex semidefinite program, written

$$\begin{array}{ll} \text{Minimize} & \mathbf{Tr}(MU) \\ \text{such that} & \mathbf{diag}(U) = 1, \; X \succeq 0 \end{array}$$

where $U \in \mathbf{H}_n$ with $n = Jp$, where p is the size of the signal.

- The complexity of solving this SDP using the algorith in Helmberg et al. [1996] is
$$O\left(J^{3.5}\, p^{3.5} \log \frac{1}{\epsilon}\right) \quad \text{and} \quad O\left(K\, J^2 p^{4.5} \log \frac{1}{\epsilon}\right)$$
for *PhaseCut* and *PhaseLift* respectively.
- Solving a generic linear system is $O(p^3)$, solving a LP is $O(p^{3.5})\ldots$
- Using first-order solvers such as TFOCS [Becker et al., 2012], based on [Nesterov, 1983], the dependence on the dimension can be further reduced, to become
$$O\left(\frac{J^3\, p^3}{\epsilon}\right) \quad \text{and} \quad O\left(\frac{K J\, p^3}{\epsilon}\right)$$
for solving *PhaseCut* and *PhaseLift* respectively, serious impact on precision.

Algorithms

Block Coordinate Method. [Wen et al., 2009]

Input: An initial $X^0 = \mathbf{I}_n$ and $\nu > 0$ (typically small). An integer $N > 1$.

for $k = 1, \ldots, N$ **do**

Pick $i \in [1, n]$.

Compute

$$x = X^k_{i^c,i^c} M_{i^c,i} \quad \text{and} \quad \gamma = x^* M_{i^c,i}$$

If $\gamma > 0$, set

$$X^{k+1}_{i^c,i} = X^{k+1*}_{i,i^c} = -\sqrt{\frac{1-v}{\gamma}}\, x$$

else

$$X^{k+1}_{i^c,i} = X^{k+1*}_{i,i^c} = 0.$$

end for

Output: A matrix $X \succeq 0$ with $\mathbf{diag}(X) = 1$.

Writing i^c the index set $\{1, \ldots, i-1, i+1, \ldots, n\}$.

Algorithms

Complexity.

- Each iteration only requires matrix vector products $O(n^2)$.
- Cost per iteration very similar to the greedy algorithm by [Gerchberg and Saxton, 1972].
- In signal applications, the matrix vector product can be computed efficiently using the **FFT**, and the cost per iteration is reduced to $O(n \log n)$.

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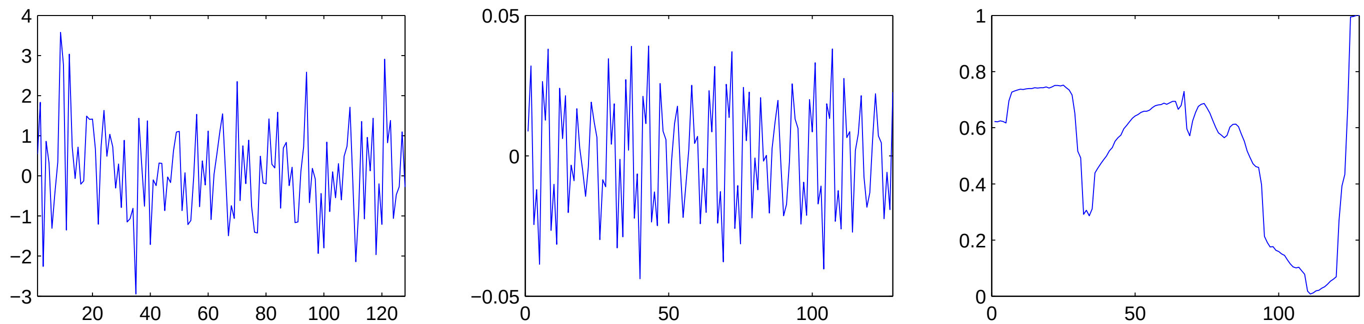
Numerical Experiments: 1D

- Three random signal classes: (a) Gaussian white noise. (b) Sum of 6 sinuoids of random frequency & random amplitudes. (c) Random scan-line of an image.



(a) (b) (c)

- The linear sampling operator A is an **oversampled Fourier transform**, multiple filterings with **random filters**, or a **wavelet transform**.
- We measure the error both in signal and in modulus

$$\epsilon(x,\tilde{x}) = \min_{c\in\mathbb{C},|c|=1} \frac{\|x - c\,\tilde{x}\|}{\|x\|} \qquad \text{and} \qquad \epsilon(|Ax|,|A\tilde{x}|) = \frac{\||Ax| - |A\tilde{x}|\|}{\|Ax\|}.$$

Numerical Experiments: 1D

	Fourier	Random Filters	Wavelets
Gerchberg-Saxton	5%	49%	0%
PhaseLift with reweighting	3%	100%	62%
PhaseCut	4%	100%	100%

Percentage of perfect reconstruction from $|Ax|$, over 300 test signals, for the three different operators A (columns) and the three algorithms (rows).

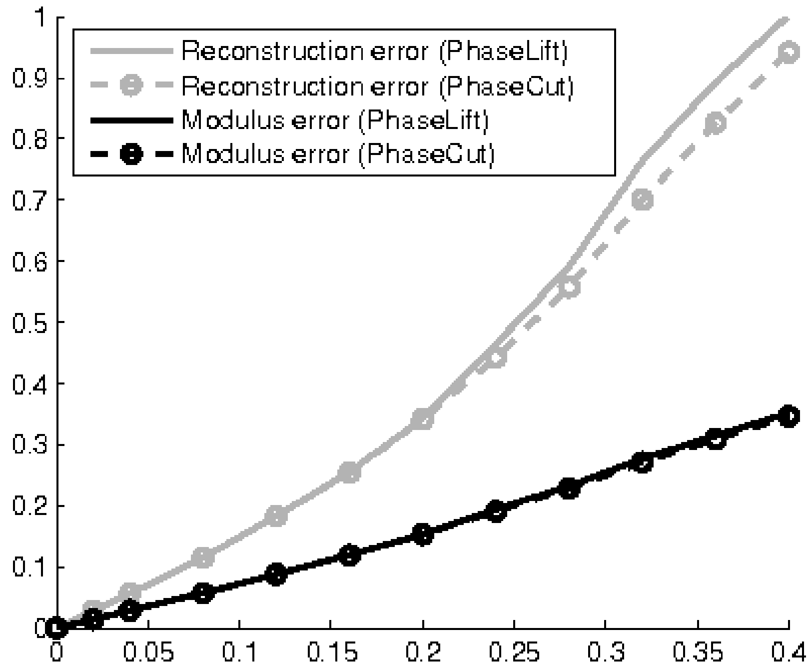
	Fourier	Random Filters	Wavelets
Gerchberg-Saxton	0.9	1.2	1.3
PhaseLift with reweighting	0.8	exact	0.5
PhaseCut	0.8	exact	exact

Average relative signal reconstruction error $\epsilon(\tilde{x}, x)$ over all test signals that are not perfectly reconstructed, for each operator A and each algorithm.

	Fourier	Random Filters	Wavelets
Gerchberg-Saxton	9.10^{-4}	0.2	0.3
PhaseLift with reweighting	5.10^{-4}	exact	8.10^{-2}
PhaseCut	6.10^{-4}	exact	exact

Average relative error $\epsilon(|A\tilde{x}|, |Ax|)$ of coefficient amplitudes, over all test signals that are not perfectly reconstructed, for each operator A and each algorithm.

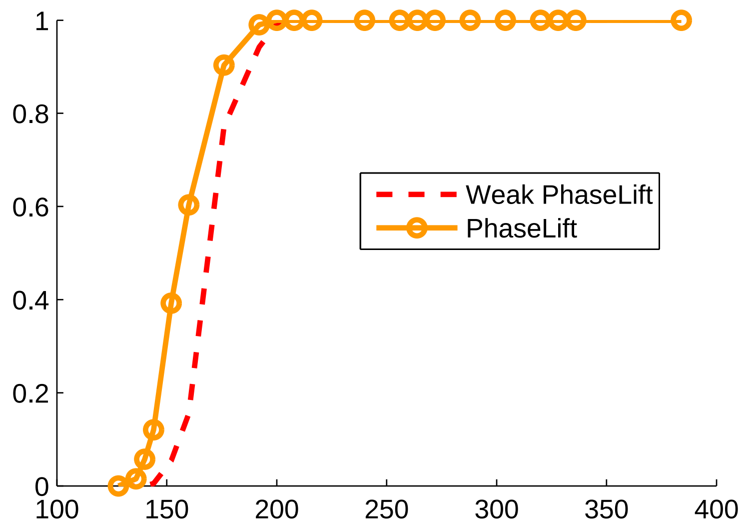
Numerical Experiments: 1D



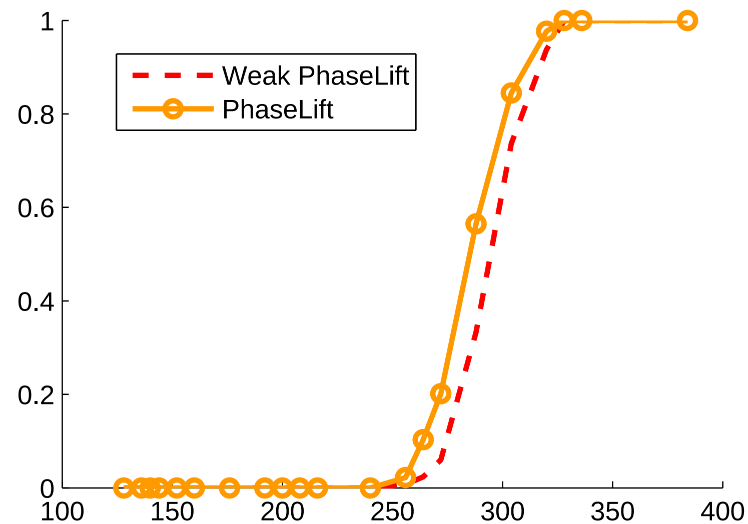
Mean performances of *PhaseLift* and *PhaseCut*, followed by some greedy iterations, for 4 gaussian random illumination filters. The x-axis represents the relative noise level, $||b_{\text{noise}}||_2/||Ax||_2$ and the y-axis the relative error on the result (signal and modulus).

Numerical Experiments: 1D

[Demanet and Hand, 2012] show that the solution to the relaxation is unique (trace minimization is unnecessary).



(a)

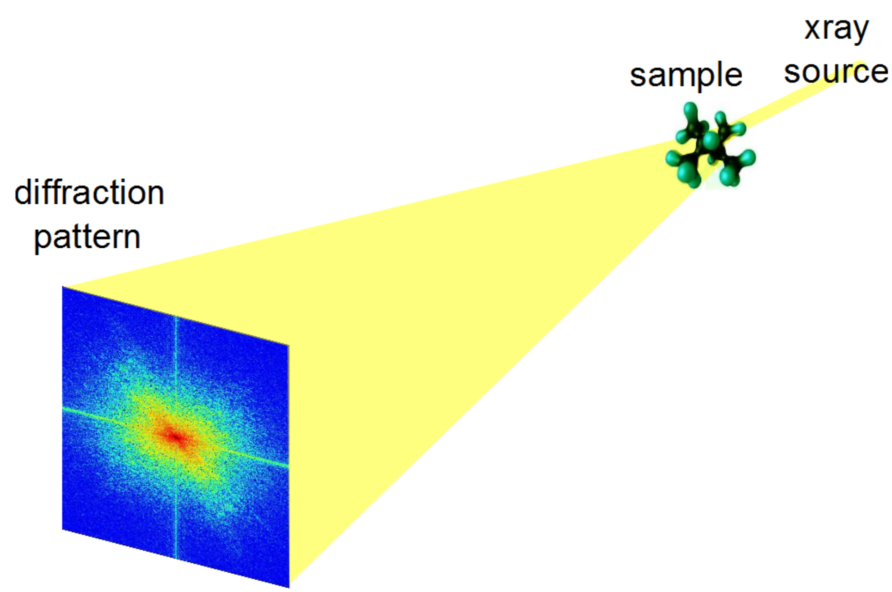


(b)

PhaseLift performance, for 64-sized signals, as a function of the number of measurements.
(a) Proportion of reconstructed signals, postprocessing using after GS iterations.
(b) Proportion of rank 1 (tight) solutions in the relaxation.

Numerical Experiments: 2D

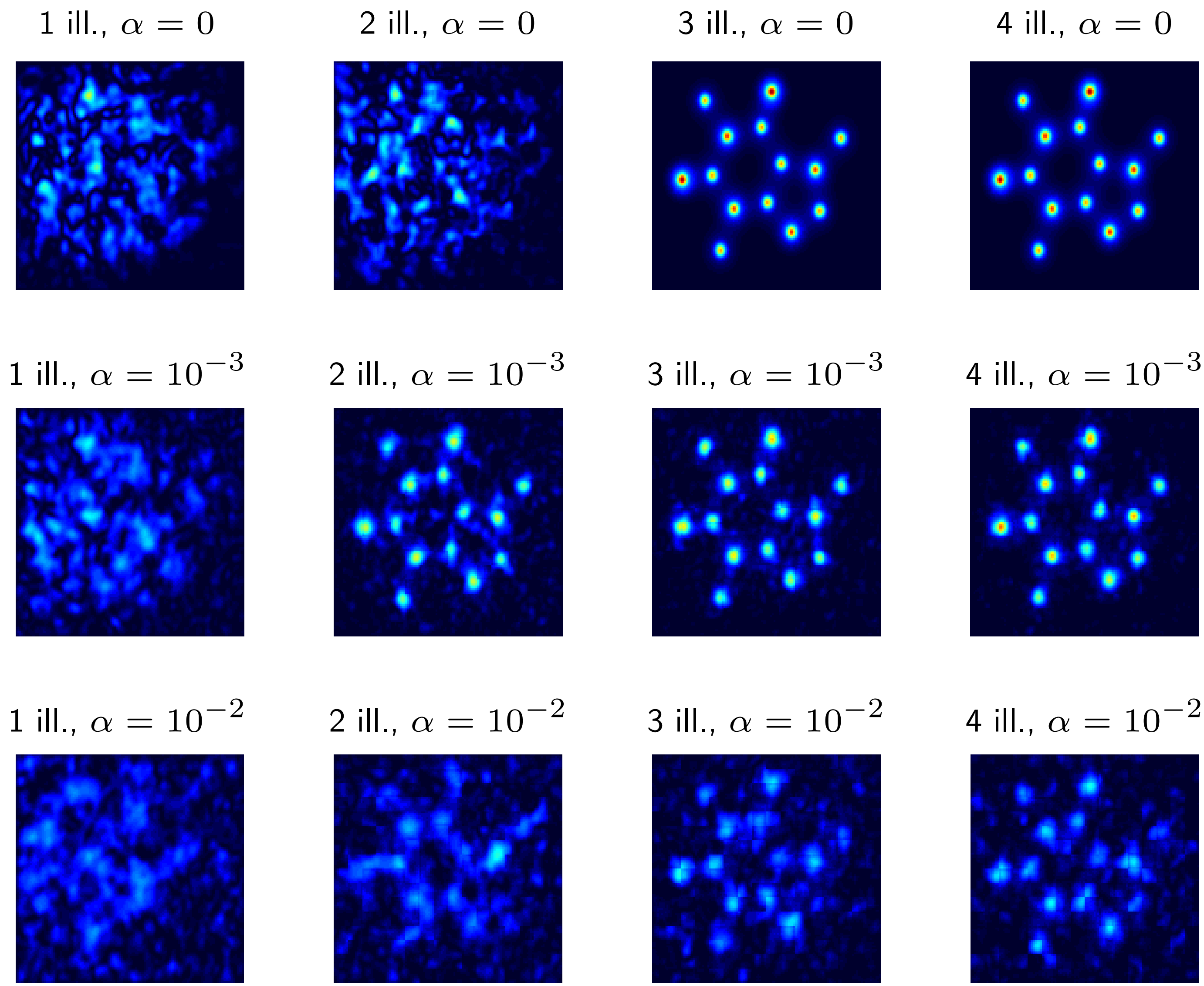
Applications in e.g. molecular imaging



(from [Candes et al., 2011b])

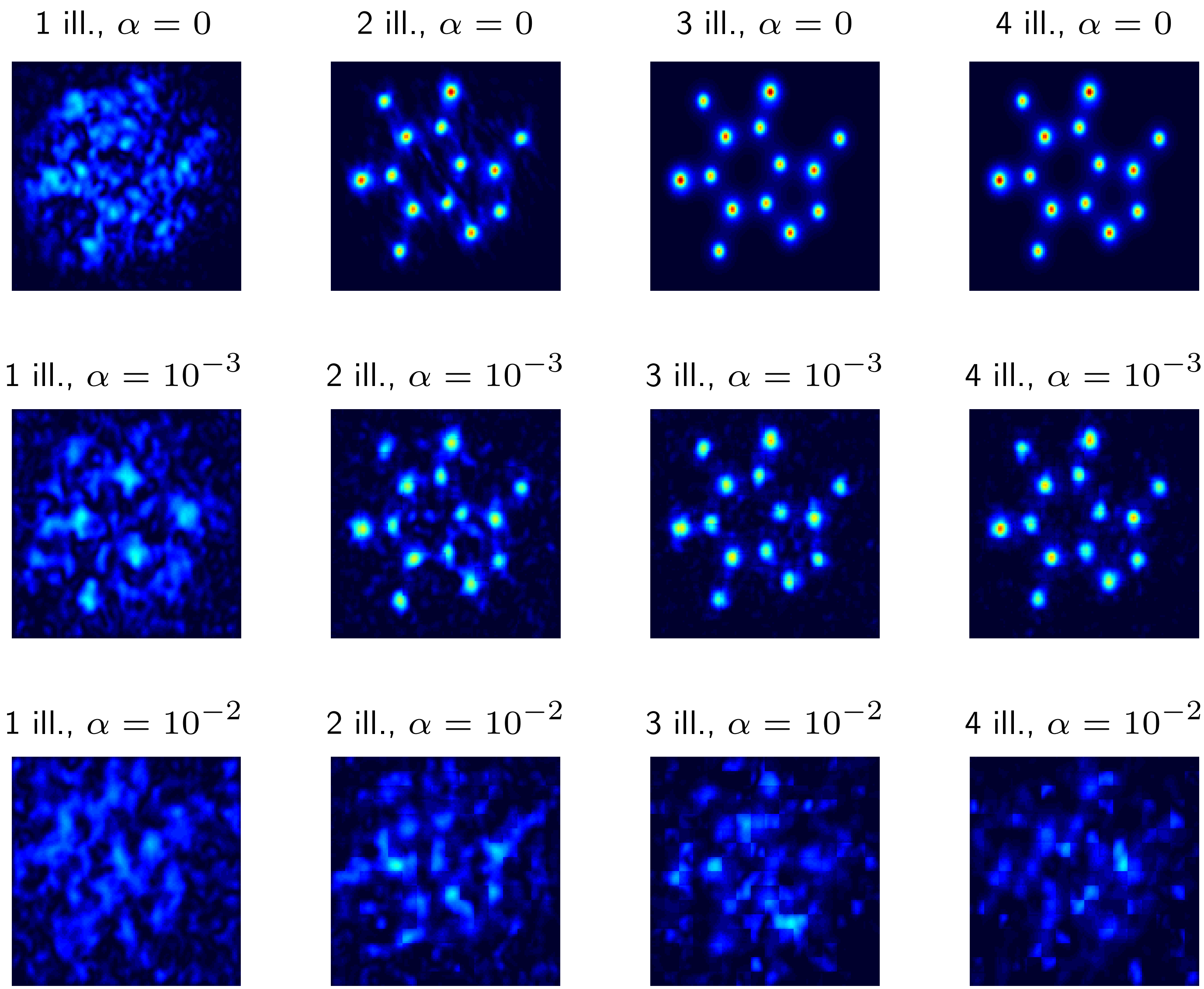
- CCD sensors only record the **magnitude** of diffracted rays, and loose the **phase**
- **Fraunhofer diffraction:** phase is required to invert the 2D Fourier transform
- Simulate diffraction using molecules from PDB and Poisson noise.

Numerical Experiments: 2D



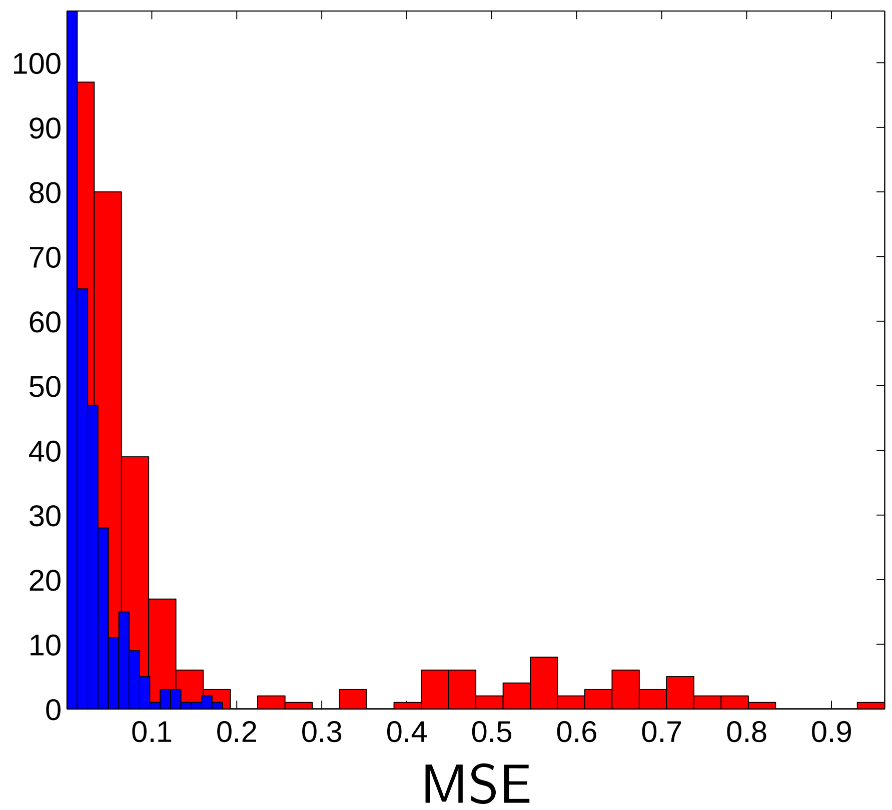
Solution of the greedy algorithm on caffeine molecule, for various values of the number of filters and noise level α.

Numerical Experiments: 2D



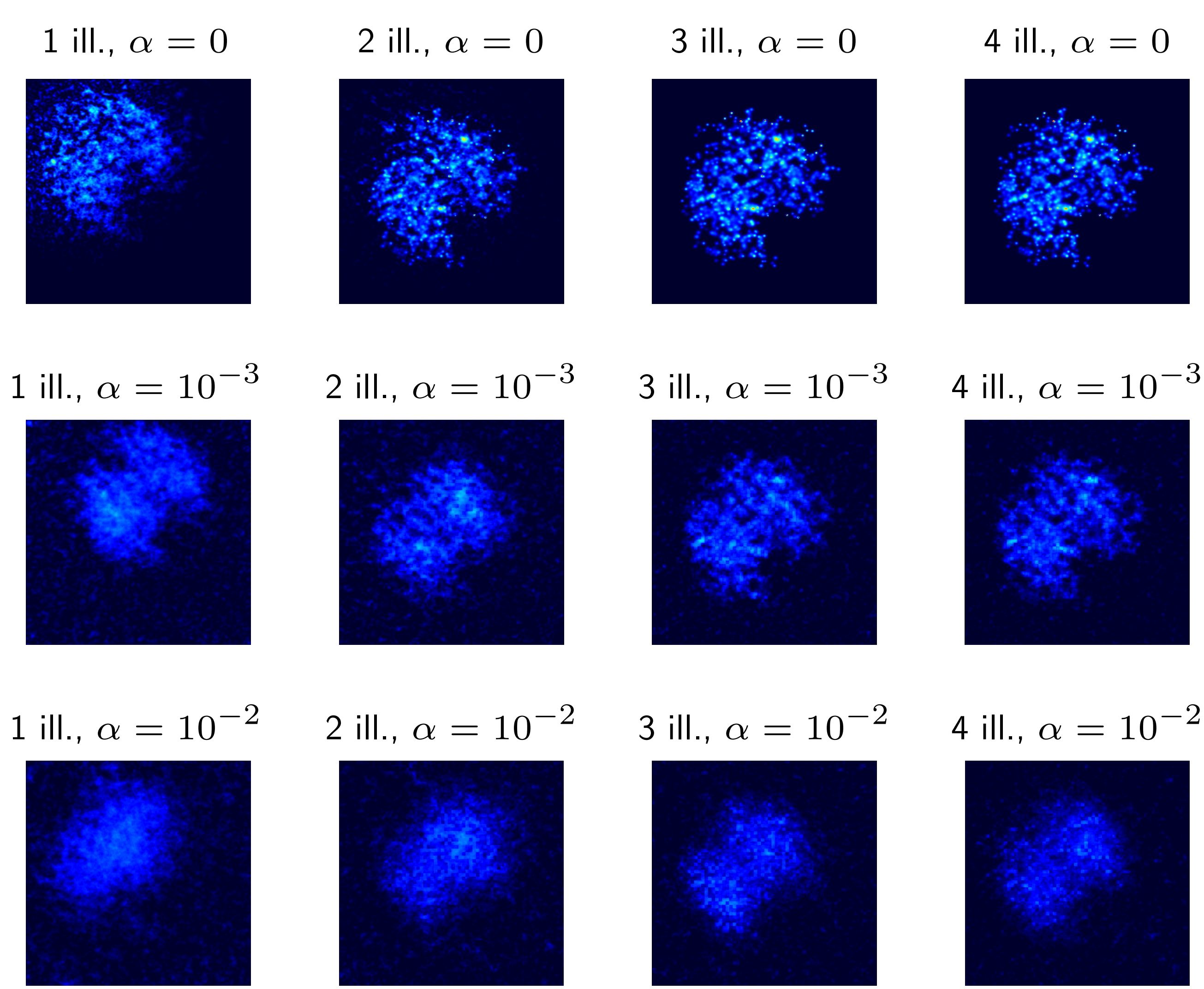
Solution of the semidefinite relaxation algorithm followed by greedy refinements, for various values of the number of filters and noise level α.

Numerical Experiments: 2D



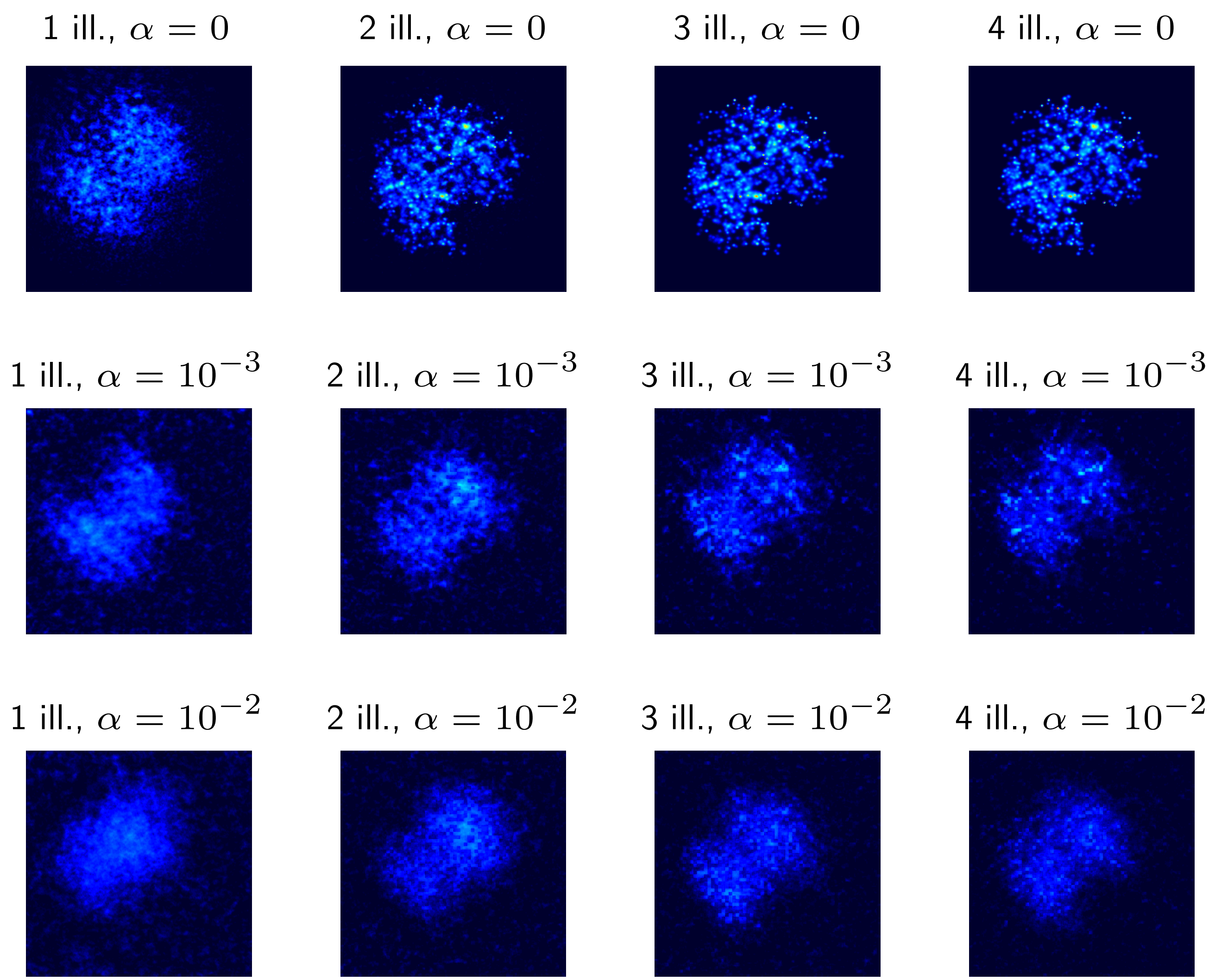
MSE between reconstructed image and true image for 2 illuminations without noise, using SDP then Fienup **(blue)**, and Fineup only **(red)**.

Numerical Experiments: 2D



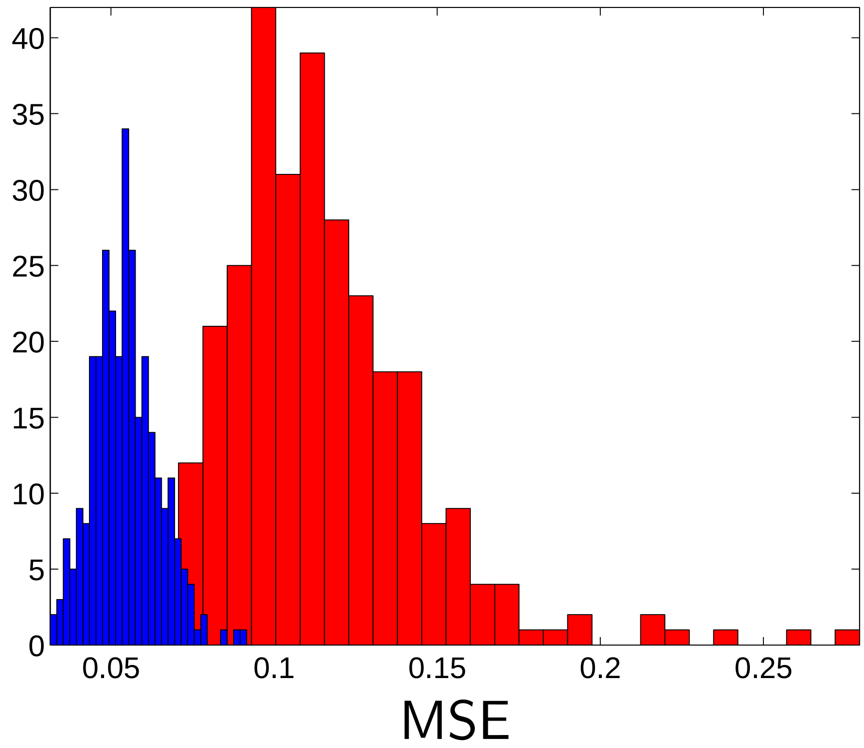
Solution of the greedy algorithm on 2LYZ (Lysozyne), for various values of the number of filters and noise level α.

Numerical Experiments: 2D



Solution of the semidefinite relaxation algorithm followed by greedy refinements on 2LYZ (Lysozyne), for various values of the number of filters and noise level α.

Numerical Experiments: 2D



MSE between reconstructed image and true image for 2 illuminations of 2LYZ without noise, using SDP then Fienup **(blue)**, and Fineup only **(red)**.

Conclusion

- Write the phase recovery problem as a MAXCUT like problem.
- Tightness properties equivalent to the matrix completion approach.
- Very fast/scalable algorithms.

Open questions. . . .

- Tightness results in the noisy case, or in the positive case?
- Is the SDP relaxation optimal?

*

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